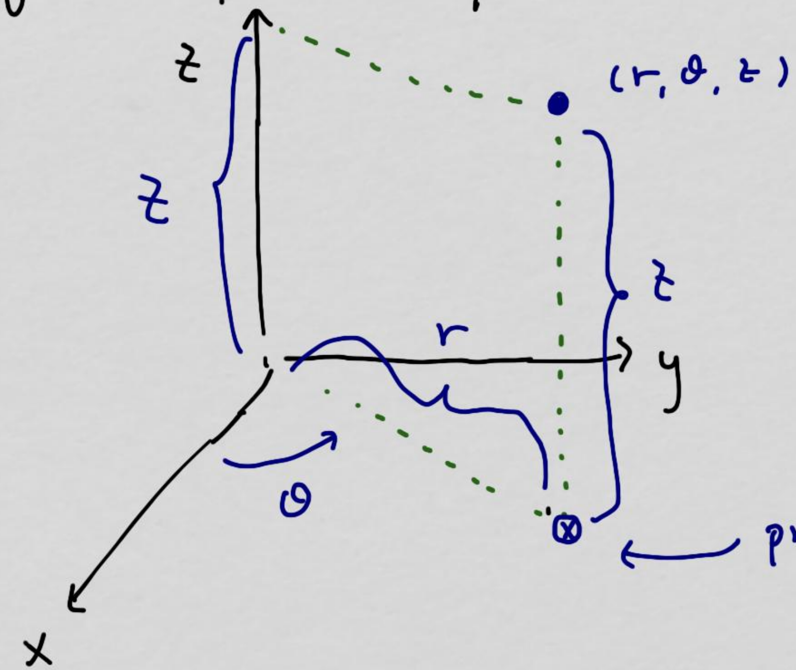
16.5 Triple Integrals in Cylindrical Coordinates

cylindrical coordinates: hybrid of polar and Cartesian coordinates

in cylindrical, a point is located by $(\underbrace{r, \theta,}_{\text{same as polar}}\ z)$ → Cartesian

in cylindrical, we have polar in x and y and Cartesian in z



projection of the point is on xy-plane described by polar and the height is Cartesian

← projection (shadow) of the point onto xy-plane

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conversion : $(x, y, z) \to (r, \theta, z)$

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\} \text{exactly like polar}$$

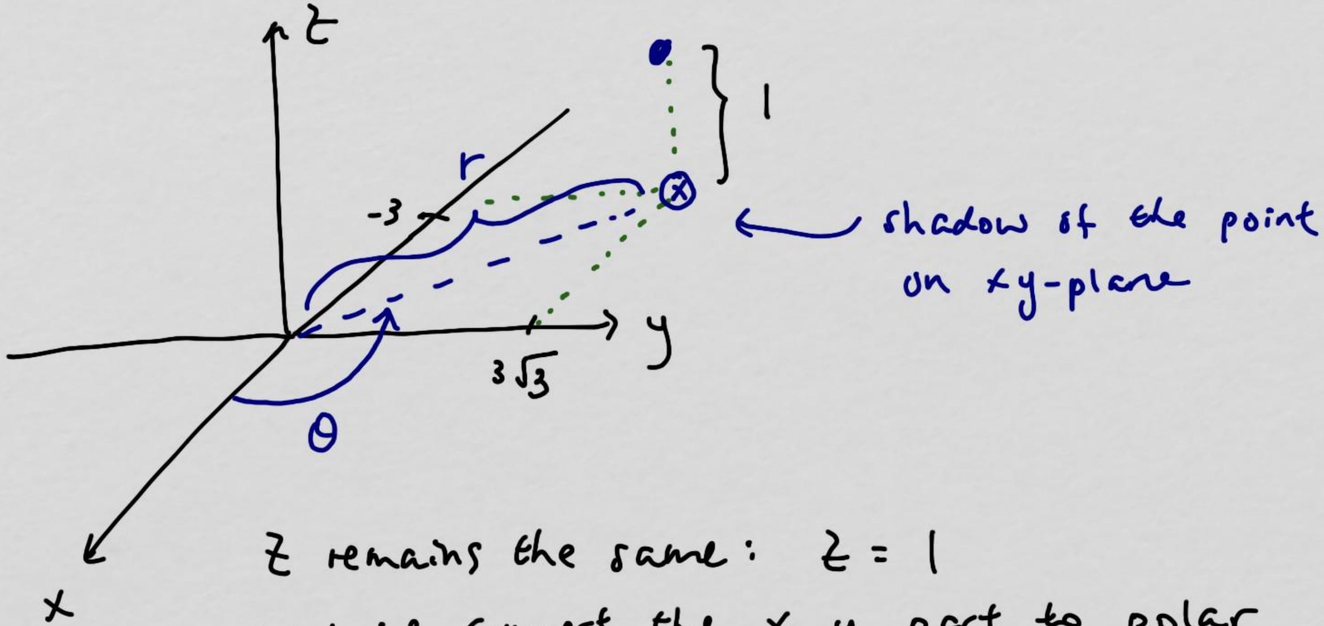
$$z = z$$

$(r, \theta, z) \to (x, y, z)$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \text{exactly like polar}$$

$$z = z$$

example express $(x, y, z) = (-3, 3\sqrt{3}, 1)$ in cylindrical



z remains the same: $z = 1$

now we convert the x, y part to polar

$r^2 = x^2 + y^2 = (-3)^2 + (3\sqrt{3})^2 = 9 + 27 = 36$

we can choose r to be positive or negative

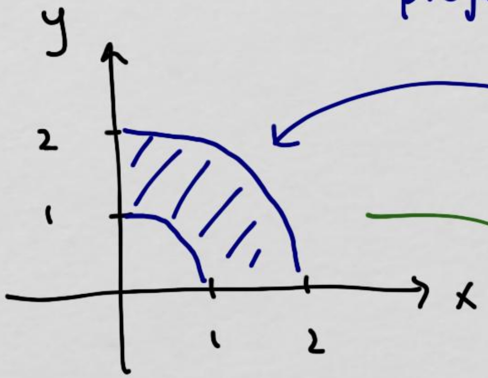
here, let's choose $r = 6$ (so we need a θ in second quadrant)

$\tan\theta = \dfrac{y}{x} = \dfrac{3\sqrt{3}}{-3} = \dfrac{\sqrt{3}}{-1} = -\dfrac{\sqrt{3}/2}{1/2} \rightarrow \theta = \dfrac{2\pi}{3}$

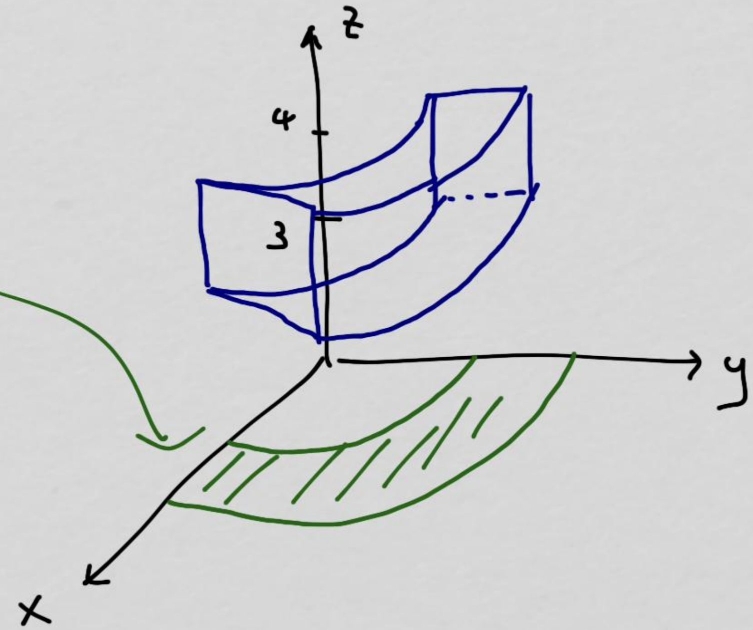
so, in cylindrical, the point is $\boxed{(r, \theta, z) = \left(6, \frac{2\pi}{3}, 1\right)}$

example Describe the space :

$$\left\{ (r, \theta, z) : \underbrace{1 \le r \le 2, \quad 0 \le \theta \le \frac{\pi}{2}}_{r \text{ and } \theta \text{ tell us what the projection (shadow) onto } xy\text{-plane}}, \quad 3 \le z \le 4 \right\}$$



the space is above this shadow



$3 \le z \le 4$

"floor" of the space

"ceiling" of the space

lift up the quarter donut to $z=3$ as floor and to $z=4$ as the ceiling

<u>example</u> $\{(r, \theta, z) : r^2 \leq z \leq 4\}$

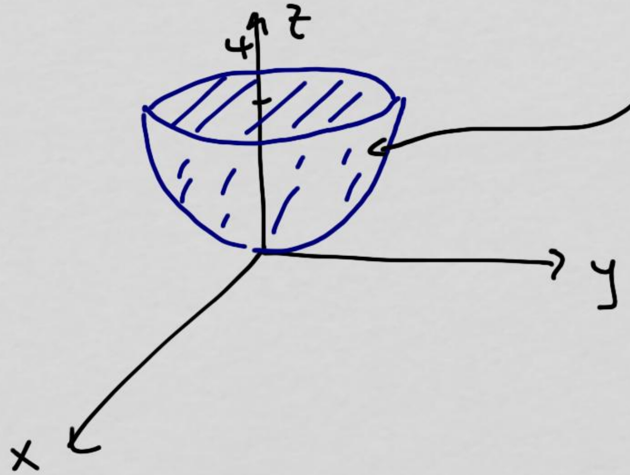
notice θ is missing $\rightarrow$ θ can be any real number

z is bounded below ("floor") by $\underbrace{z = r^2}_{\text{shape?}}$

easier to identify in Cartesian: $z = x^2 + y^2$

paraboloid

z is bounded above ("ceiling") by $z = 4$ plane



the solid described by the expression above

Cylindrical is particularly effective in integration if the volume/space is a cylinder or cylinder-like (paraboloid, cone)

or when polar is good (circles or circle-like shapes)

<u>example</u>
$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$$

terrible integral in Cartesian!

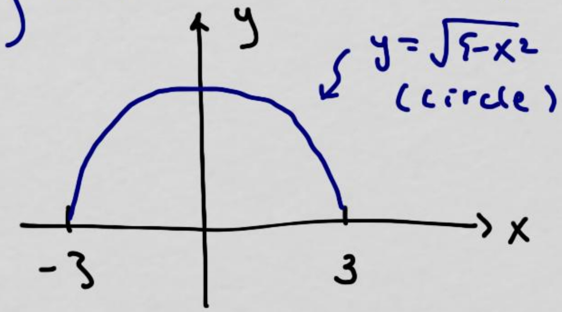
notice the integration bounds for y and z suggest we are dealing with circle and paraboloid (round shapes/edges)

so, cylindrical should lead to an easier integral

$$\left.\begin{aligned}-3 &\le x \le 3\\ 0 &\le y \le \sqrt{9-x^2}\end{aligned}\right\}$$ projection/shadow of volume on xy-plane

$y=\sqrt{9-x^2}$

$y^2 = 9-x^2 \to x^2+y^2=9$

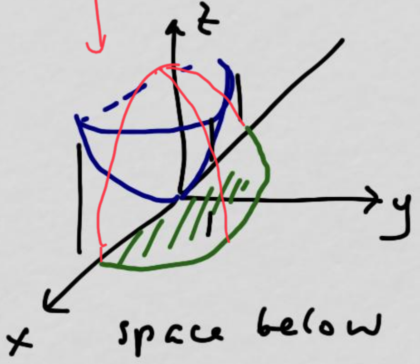


$$\Longrightarrow \boxed{\begin{aligned}0 &\le r \le 3\\ 0 &\le \theta \le \pi\end{aligned}}$$

now look at z: $\quad 0 \le z \le \underbrace{9 - x^2 - y^2}_{\substack{\text{express in polar / cylindrical} \\ 9 - r^2}}$

I drew the paraboloid upside down, sorry!

therefore, $\boxed{\underbrace{0}_{\text{xy-plane}} \le z \le \underbrace{9 - r^2}_{\text{paraboloid}}}$ the volume looks like



space below paraboloid above xy-plane

old integral
$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \underbrace{\sqrt{x^2+y^2}}_{\text{becomes } r} \; dz \, \underbrace{dy \, dx}_{\text{becomes } r\,dr\,d\theta}$$

becomes
$$\int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r \; dz \, r \, dr \, d\theta = \int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r^2 \, dz \, dr \, d\theta$$

$\theta \quad r \quad z$

$$= \cdots = \boxed{\frac{162\pi}{5}}$$

Much easier!

<u>example</u>

$$\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\, dx\, dz$$

again, terrible integral in Cartesian

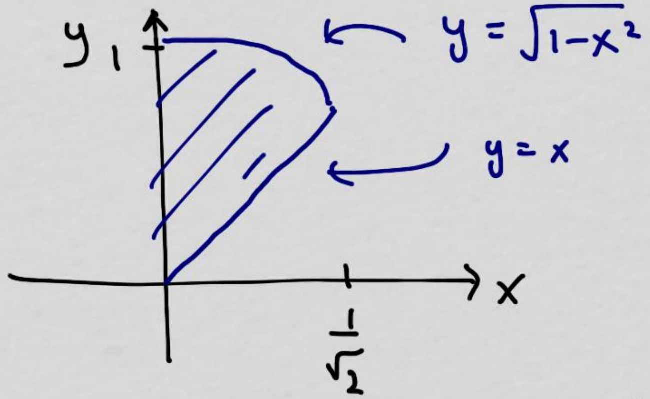
the $\sqrt{1-x^2}$ limit in y suggests a circle $\longrightarrow$ cylindrical

$$0 \le z \le 4$$

$$\left.\begin{aligned} 0 &\le x \le 1/\sqrt{2} \\ x \le y &\le \sqrt{1-x^2} \end{aligned}\right\}$$

x ↓ line

$\sqrt{1-x^2}$ ↓ circle radius 1

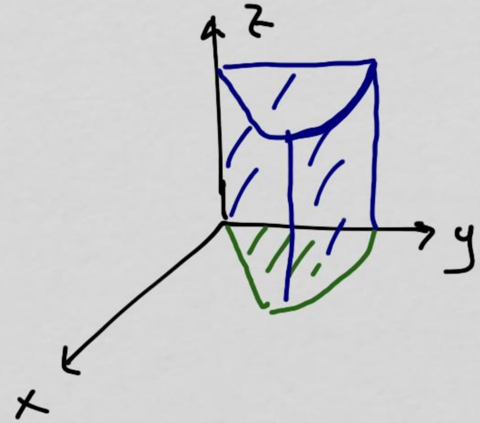


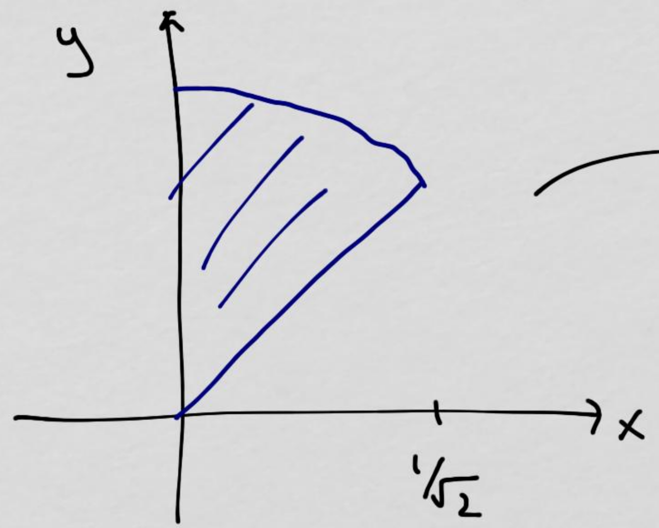
this is the shadow of the volume

$$0 \le z \le 4$$

floor ↙ ↘ ceiling

both are planes





$$\boxed{\begin{array}{c} 0 \le r \le 1 \\ \frac{\pi}{4} \le \theta \le \frac{\pi}{4} \end{array}} \qquad \boxed{0 \le z \le 4}$$

from the line $y = x$ which has slope of 1 so divides first quadrant into 2 equal parts each with 45°

the integrand $e^{-x^2-y^2}$ becomes $e^{-(x^2+y^2)} = e^{-r^2}$

old integral $$\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2} \underbrace{dy\,dx}_{r\,dr\,d\theta}\,dz$$

becomes $$\int_{\underset{z}{0}}^4 \int_{\underset{\theta}{\pi/4}}^{\pi/2} \int_{\underset{r}{0}}^1 e^{-r^2}\, r\,dr\,d\theta\,dz = \cdots = \boxed{\frac{\pi}{2}\left(1-e^{-1}\right)}$$

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triple integrals can be used to find volume : $\iiint_D 1\, dV = V$

in general, $\iiint_D f(x,y,z)\, dV$ can represent a lot of things

for example, if $f(x,y,z)$ is the density, then $\iiint_D f(x,y,z)\, dV$ is the accumulation of density through the space which gives us the mass.

<u>example</u> Find the mass of the solid bounded above by $x^2+y^2+z^2 = 4$ and bounded below by $z = \sqrt{x^2+y^2}$ with density $\rho(x,y,z) = z$

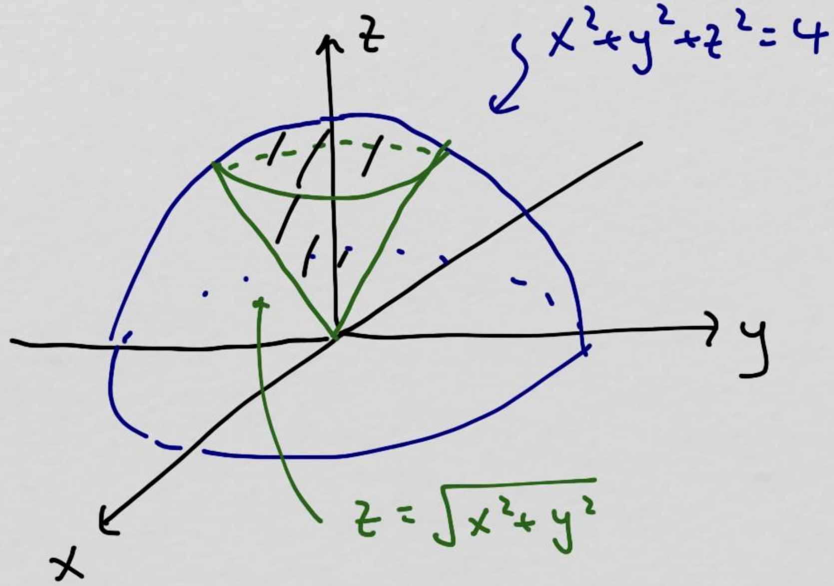
Greek letter "rho"

ceiling of this shape ("bound above by") is $x^2+y^2+z^2=4$

sphere of radius 2

floor of this shape ("bounded below by") is $z=\sqrt{x^2+y^2}$

cone



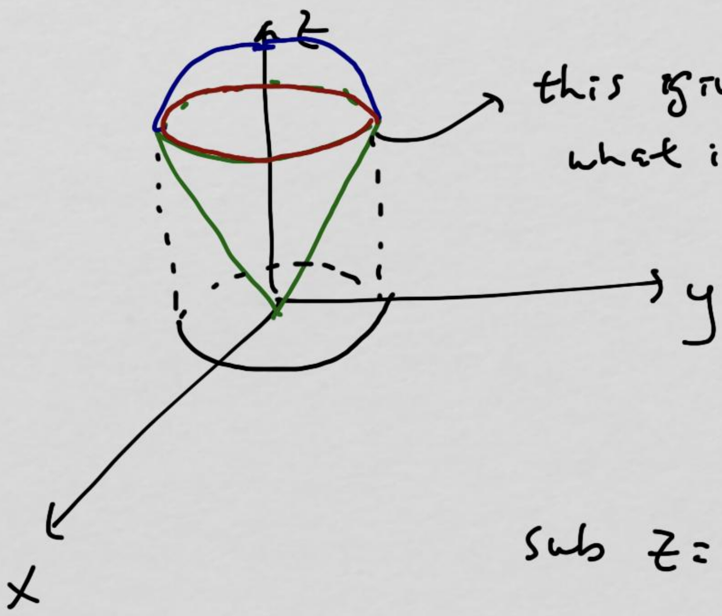
so, we want mass of

with density $\underbrace{\rho(x,y,z) = z}$

the higher the z, the denser and therefore more mass obviously, the shape is not good for Cartesian

let's do this in cylindrical

we use the projection / shadow of the shape onto xy-plane to find bounds for r and θ



this gives us the shadow

what is its radius? This circle is the intersection of $x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$

so it has the same z on both surfaces

sub $z=\sqrt{x^2+y^2}$ into $x^2+y^2+z^2=4$

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=4$$

$$2x^2+2y^2=4 \longrightarrow x^2+y^2=2$$

so, the shadow on xy-plane is a $\underset{\text{whole}}{\wedge}$ circle of radius $\sqrt{2}$

$$\boxed{\begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le \sqrt{2} \end{array}}$$

z is bounded above by $x^2+y^2+z^2=4$

$$r^2+z^2=4 \longrightarrow z=\sqrt{4-r^2}$$

and bounded below by $z=\sqrt{x^2+y^2} \longrightarrow z=r$

so, $\boxed{r \le z \le \sqrt{4-r^2}}$

now integrate the density $\rho(x,y,z)=z$ which remains z in cylindrical

$$\underset{\theta}{\int_0^{2\pi}} \underset{r}{\int_0^{\sqrt{2}}} \underset{z}{\int_r^{\sqrt{4-r^2}}} z \; dz\, r\, dr\, d\theta = \cdots = \boxed{2\pi}$$