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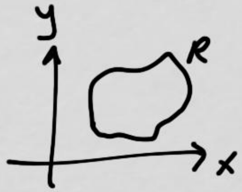
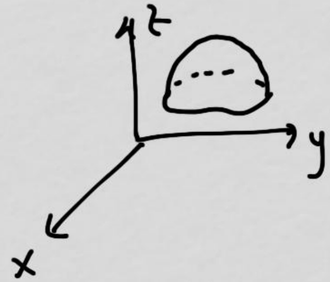
16.6 Integrals for Mass Calculations

we know how to calculate mass: $\iint_R \rho(x,y)\,dA$



$$\iiint_D \rho(x,y,z)\,dV$$



in 16.6 we will find the center of mass of 2D and 3D shapes

first, we need to know the quantity called moment - it's the rotational equivalent of momentum and force. It measures how strong the rotation is about an axis

moment = mass · distance from the rotational axis

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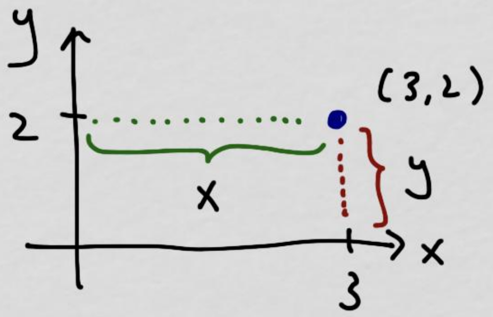

point has a mass of 4

the moment about the y-axis is

$$M_y = \text{mass} \cdot \underbrace{\text{distance from the y-axis}}_{\text{is actually } x}$$

$$M_y = (4)(3) = 12$$

likewise, the moment about the x-axis is

$$M_x = \text{mass} \cdot \underbrace{\text{distance from x-axis}}_{\text{is actually } y}$$

$$M_x = (4)(2) = 8$$

here, $M_y > M_x$, so this means the rotation about the y-axis is stronger than the rotation about the x-axis

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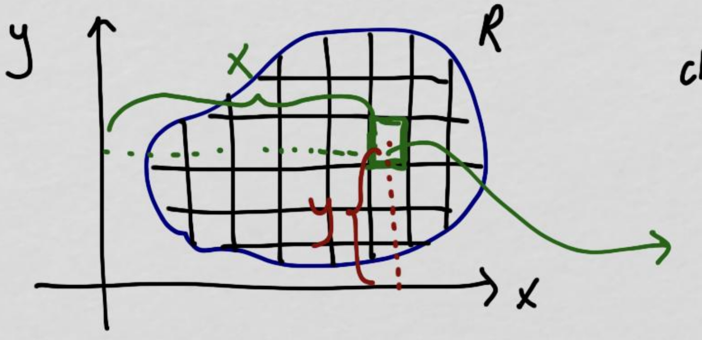
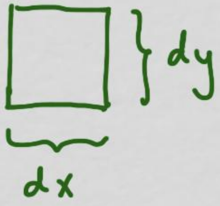
now we can use the same idea to find the moments for a plate whose shape is described by the region R and has density $\rho(x,y)$



chop into little pieces



mass of this little piece is

$$\rho(x,y)\,dA = \rho(x,y)\,dy\,dx$$

the moment of this small piece about the y-axis is

$$M_y = \text{mass} \cdot \underbrace{\text{distance from } y\text{-axis}}_{x} = \rho(x,y)\,dA \cdot x = x\,\rho(x,y)\,dA$$

Similarly, the moment about the x-axis is

$$M_x = \text{mass} \cdot \underbrace{\text{distance from } x\text{-axis}}_{y} = \rho(x,y)\,dA \cdot y = y\,\rho(x,y)\,dA$$

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now we accumulate all the small piece by integration to find M_x and M_y for the entire plate

$$\boxed{\begin{aligned} M_x &= \iint_R y\,\rho(x,y)\,dA \\ M_y &= \iint_R x\,\rho(x,y)\,dA \end{aligned}}$$

now we have moments, we can find center of mass

center of mass – the location $(\bar{x}, \bar{y})$ where if we place a point mass

"x bar"

that has the same mass as the plate, the point mass would have the same moments as the plate

(in other words, if we shrink the plate into one point w/ the same mass, where do we put it so it would keep the same moments as the plate?)

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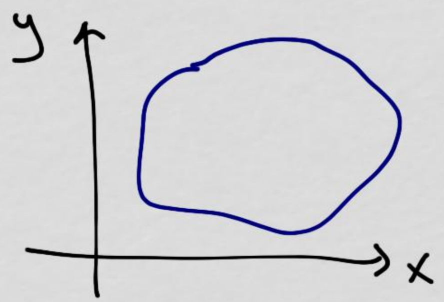


shrink into a point $\longrightarrow$

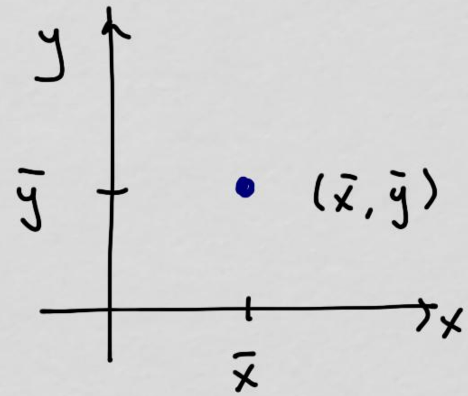


plate has

$$M_x = \iint_R y \rho \, dA \quad \xleftrightarrow{\text{equal}}$$

$$M_y = \iint_R x \rho \, dA \quad \xleftrightarrow{\text{equal}}$$

point has

$$M_x = m \cdot \bar{y}$$

$$M_y = m \cdot \bar{x}$$

$$m = \iint_R \rho \, dA$$

so,

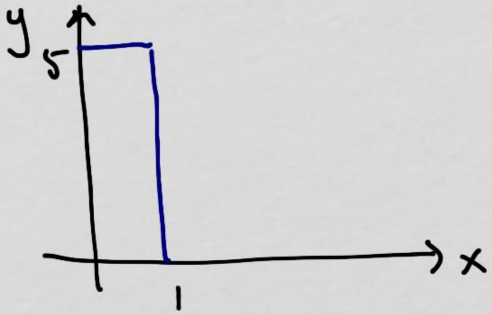
$$M_x : \iint_R y \rho \, dA = m \cdot \bar{y}$$

$$M_y : \iint_R x \rho \, dA = m \cdot \bar{x}$$

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iint_R x \rho \, dA \\ \bar{y} &= \frac{1}{m} \iint_R y \rho \, dA \\ m &= \iint_R \rho \, dA \end{aligned}}$$

<u>example</u> $R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 5\}$

$$\rho(x,y) = 2e^{-\frac{1}{2}y}$$



mass: $$m = \iint_R \rho\, dA = \int_0^1 \int_0^5 2e^{-\frac{1}{2}y}\, dy\, dx$$

$$= \cdots = 4(1-e^{-5/2})$$

$$\bar{x} = \frac{1}{m}\iint_R x\rho\, dA = \frac{1}{4(1-e^{-5/2})}\int_0^1 \int_0^5 x \cdot 2e^{-\frac{1}{2}y}\, dy\, dx$$

$$= \frac{1}{4(1-e^{-5/2})} \cdot 2(1-e^{-5/2}) = \boxed{\frac{1}{2}}$$

$$\bar{y} = \frac{1}{m}\iint_R y\rho\, dA = \frac{1}{4(1-e^{-5/2})}\int_0^1 \underbrace{\int_0^5 y \cdot 2e^{-\frac{1}{2}y}\, dy}_{\text{int. by parts}}\, dx$$

$$= \cdots = \frac{-14e^{-5/2}+4}{2(1-e^{-5/2})} \approx \boxed{1.6}$$

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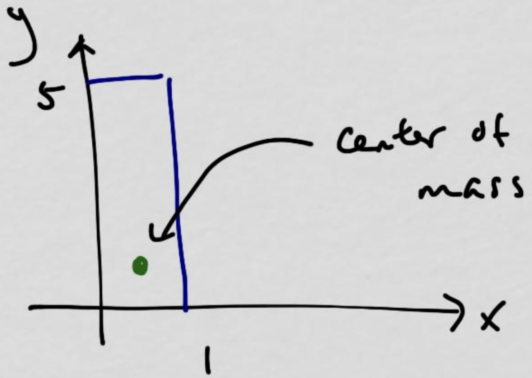
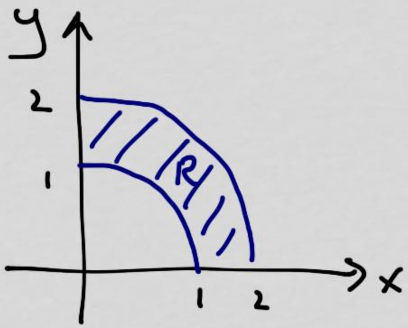


$\bar{x} = 1/2$

$\bar{y} \approx 1.6$

in general, unless the density is constant, the center of mass and the geometric center (centroid) are not the same point

<u>example</u> R : between circles of radii 1 and 2 centered at origin in the first quadrant. $\rho(x,y) = \sqrt{x^2+y^2}$



circles, therefore we use polar coordinates

$0 \le \theta \le \pi/2$

$1 \le r \le 2$

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$$m = \iint_R \rho \, dA \qquad \rho = \sqrt{x^2+y^2} = \sqrt{r^2} = r$$

$$= \int_0^{\pi/2} \int_1^2 \underbrace{r}_{\rho} \, \underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\pi/2} \int_1^2 r^2 \, dr\,d\theta = \dots = \frac{7\pi}{6}$$

$$\bar{y} = \frac{1}{m} \iint_R y\rho \, dA = \frac{1}{7\pi/6} \int_0^{\pi/2} \int_1^2 \underbrace{r\sin\theta}_{y} \cdot \underbrace{r}_{\rho} \cdot \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \frac{6}{7\pi} \int_0^{\pi/2} \int_1^2 r^3 \sin\theta \, dr\,d\theta = \dots = \frac{45}{14\pi} \approx 1.023$$

$$\bar{x} = \frac{1}{m} \iint_R x\rho \, dA = \frac{1}{7\pi/6} \int_0^{\pi/2} \int_1^2 \underbrace{r\cos\theta}_{x} \cdot \underbrace{r}_{\rho} \cdot \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \frac{6}{7\pi} \int_0^{\pi/2} \int_1^2 r^3 \cos\theta \, dr\,d\theta = \dots = \frac{45}{14\pi} \approx 1.023$$

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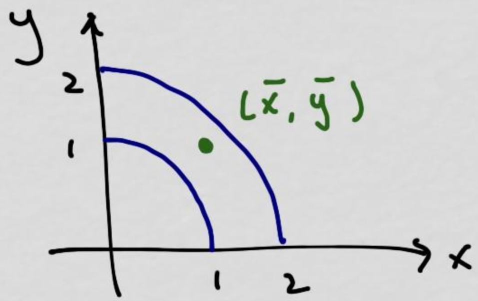

$\bar{x} \approx 1.023$

$\bar{y} \approx 1.023$

extending to 3D is easy: $\underbrace{dA \to dV}_{\text{double integral becomes a triple integral}}$, $\rho(x,y) \to \rho(x,y,z)$

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iiint_D x\,\rho(x,y,z)\,dV \\ \bar{y} &= \frac{1}{m} \iiint_D y\,\rho(x,y,z)\,dV \\ \bar{z} &= \frac{1}{m} \iiint_D z\,\rho(x,y,z)\,dV \\ m &= \iiint_D \rho(x,y,z)\,dV \end{aligned}}$$

$\iiint_D x\rho\,dV$: moment about the yz-plane, M_{yz}

$\iiint_D y\rho\,dV$: moment about the xz-plane, M_{xz}

$\iiint_D z\rho\,dV$: moment about the xy-plane, M_{xy}

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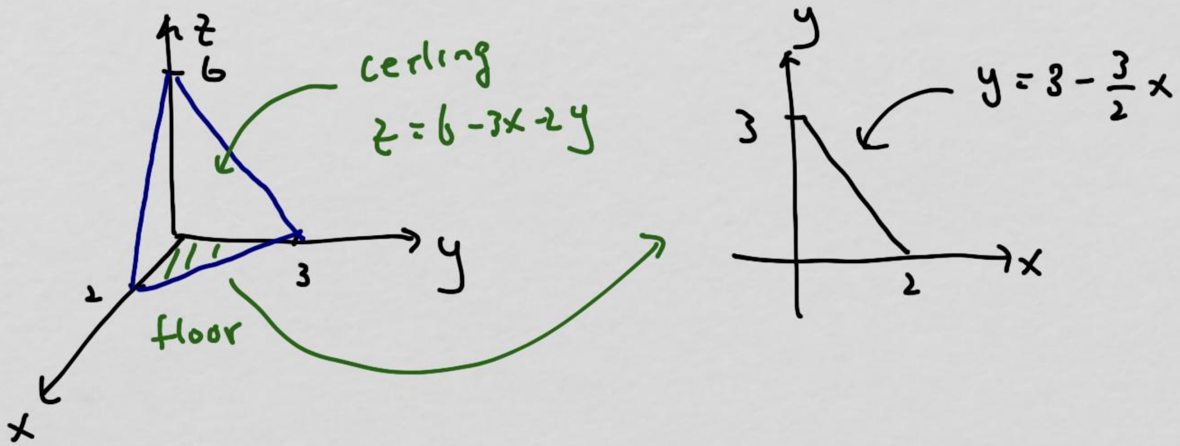
<u>example</u> Find the center of mass of the solid bounded by $3x+2y+z=6$ and by the coordinate planes.

$\rho(x,y,z) = 1+x$



so, the bounds are

$$0 \le x \le 2$$
$$0 \le y \le 3 - \frac{3}{2}x$$
$$0 \le z \le 6 - 3x - 2y$$

mass: $m = \iiint_D \rho(x,y,z)\,dV = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} (1+x)\,dz\,dy\,dx = \cdots = 9$

$$\bar{x} = \frac{1}{m}\iiint_D x\rho\, dV = \frac{1}{9}\int_0^2\int_0^{3-\frac{3}{2}x}\int_0^{6-2x-3y} x\,\underbrace{(1+x)}_{\rho}\, dz\,dy\,dx = \cdots = \frac{3}{5}$$

$\bar{y}, \bar{z}$ are done the same way $\quad \bar{y} = \cdots = \frac{7}{10} \qquad \bar{z} = \frac{7}{5}$

a related quantity, called the moment of inertia (or angular mass) is the rotational equivalent of mass, and the higher it is, the harder it is to rotate it.

$$\underbrace{I_{axis}}_{\text{moment of inertia about an axis}} = \iint_R \rho(x,y)\, d^2\, dA \qquad (2D)$$

moment of inertia about an axis

(I_x is moment of inertia about the x-axis)

↳ d: distance of a point in R to the axis of rotation

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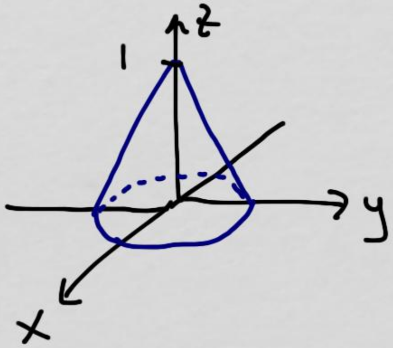
$$I_{axis} = \iiint_D \rho(x,y,z)\, d^2\, dV \qquad (3D)$$

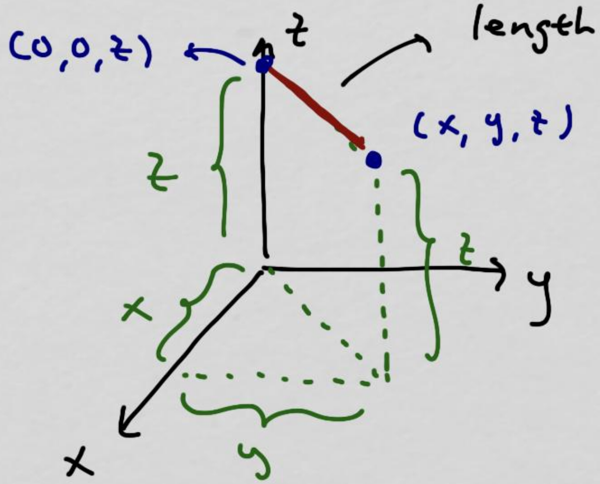
<u>example</u> Find the moment of inertia about the z-axis of the solid bounded above by $z = 1 - \sqrt{x^2+y^2}$ and bounded below by $z = 0$ with $\rho(x,y,z) = z$



the shape suggests that cylindrical is good

$$I_z = \iiint_D \rho(x,y,z)\, d^2\, dV$$

d is a bit of work:



length of this is the distance of (x,y,z) to the z-axis (the "d" we want)

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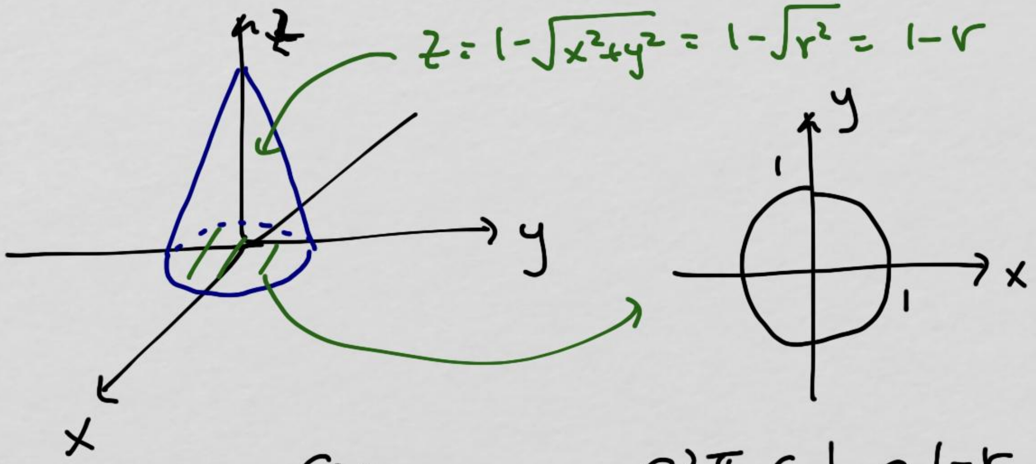
so, $d = \sqrt{(x-0)^2+(y-0)^2+(z-z)^2} = \sqrt{x^2+y^2}$



$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$0 \le z \le 1-r$

$$I_z = \iiint_D \rho d^2 \, dV = \int_0^{2\pi}\int_0^1\int_0^{1-r} z\,(r)^2 \, \overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$z \downarrow \rho$

$(r) \downarrow$ $d = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r} r^3 z \, dz\,dr\,d\theta = \dots = \boxed{\frac{\pi}{60}}$$