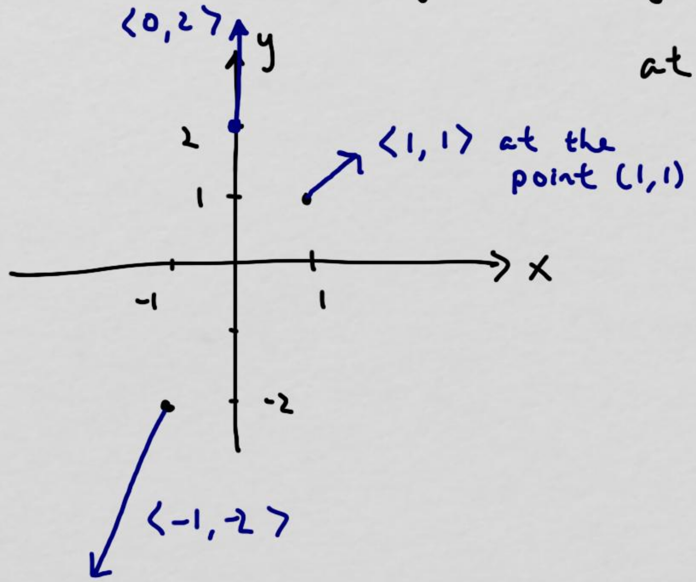
17.1 Vector Fields

a vector field is a function that assigns a vector to a point

for example, $\vec{F}(x,y) = \langle x, y \rangle$



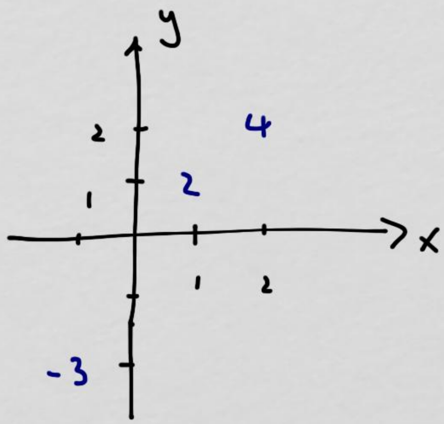
at $(1,1)$, $\vec{F}(1,1) = \langle 1, 1 \rangle$

at $(0,2)$, $\vec{F}(0,2) = \langle 0, 2 \rangle$

at $(-1,-2)$, $\vec{F}(-1,-2) = \langle -1, -2 \rangle$

in contrast, the functions we have been working with recently are all

scalar fields : functions that assign scalars to points for example, $f(x,y) = x+y$



at $(1,1)$ $f(1,1) = 2$ ← scalar at $(1,1)$

at $(2,2)$ $f(2,2) = 4$

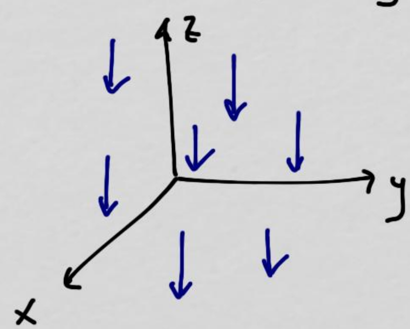
at $(-1,-2)$ $f(-1,-2) = -3$

the temperature of every point in the room you are in is also a scalar field – each point has a temperature associated with it (scalar)

the acceleration due to gravity in your room is a vector field – every point has an acceleration vector due to gravity

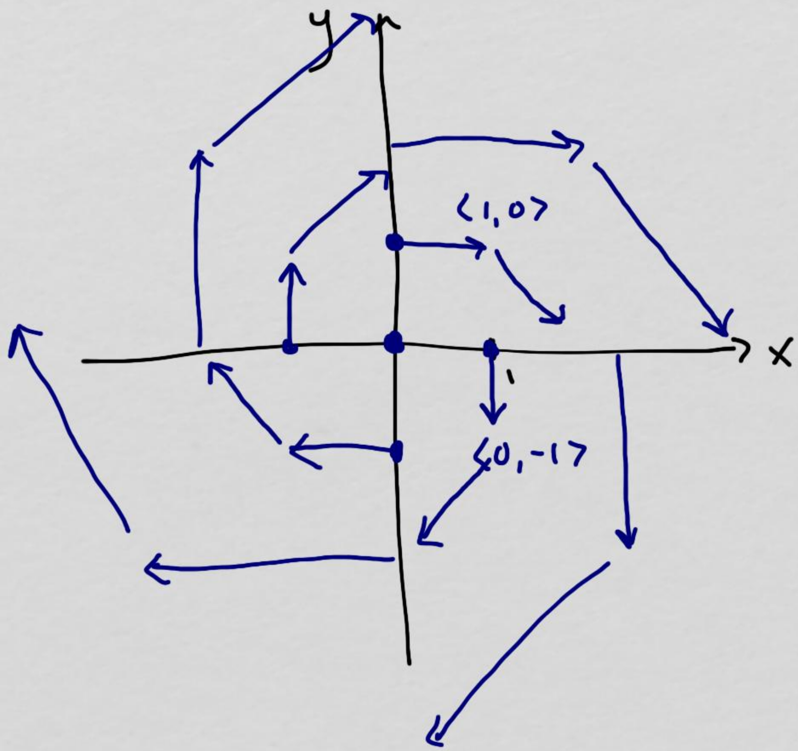
$$\vec{F}(x,y,z) = \langle 0, 0, -g \rangle$$

acceleration due to gravity



let's sketch a few more vector fields

example $\vec{F}(x,y) = \langle y, -x \rangle$



$\vec{F}(0,0) = \langle 0, 0 \rangle$ zero vector

$\vec{F}(1,0) = \langle 0, -1 \rangle$

$\vec{F}(0,1) = \langle 1, 0 \rangle$

$\vec{F}(-1,0) = \langle 0, 1 \rangle$

$\vec{F}(0,-1) = \langle -1, 0 \rangle$

$\vec{F}(1,1) = \langle 1, -1 \rangle$

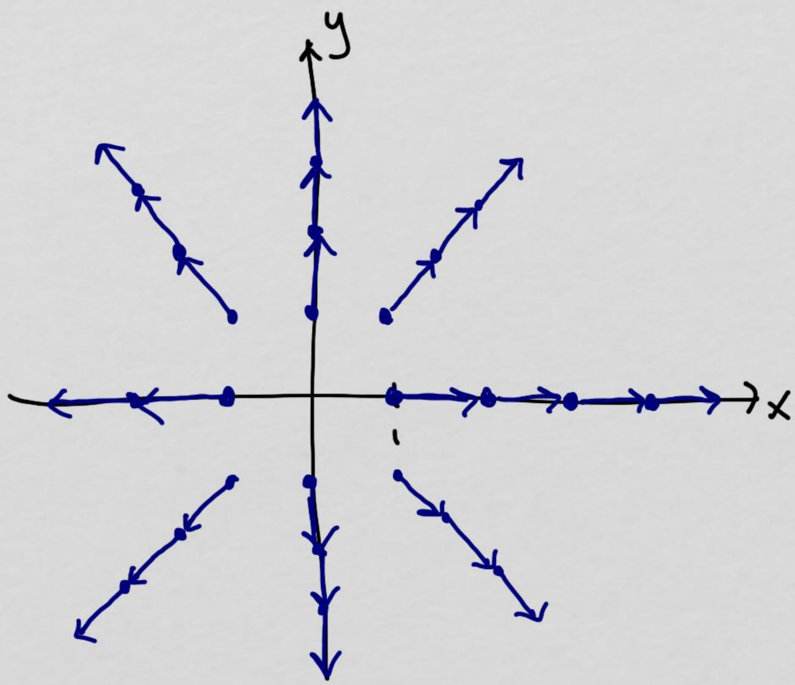
$\vec{F}(1,-1) = \langle -1, -1 \rangle$

note the farther away the point is from the origin, the longer the associated vector is

$\vec{F}(2,2) = \langle 2, -2 \rangle$

clockwise spiral

<u>example</u> $\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$



we can pick points like we did before, or we can analyze the vector field more efficiently

$$\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$= \underbrace{\frac{1}{\sqrt{x^2+y^2}} \underbrace{\langle x, y \rangle}_{\text{has magnitude } \sqrt{x^2+y^2}}}_{\text{unit vector}}$$

at $(1,0)$, $\langle x,y\rangle = \langle 1,0\rangle$

at $(2,0)$, unit vector in the direction of $\langle 2,0\rangle$

every single point has a unit vector in the direction of $\langle x, y\rangle$ except at the origin

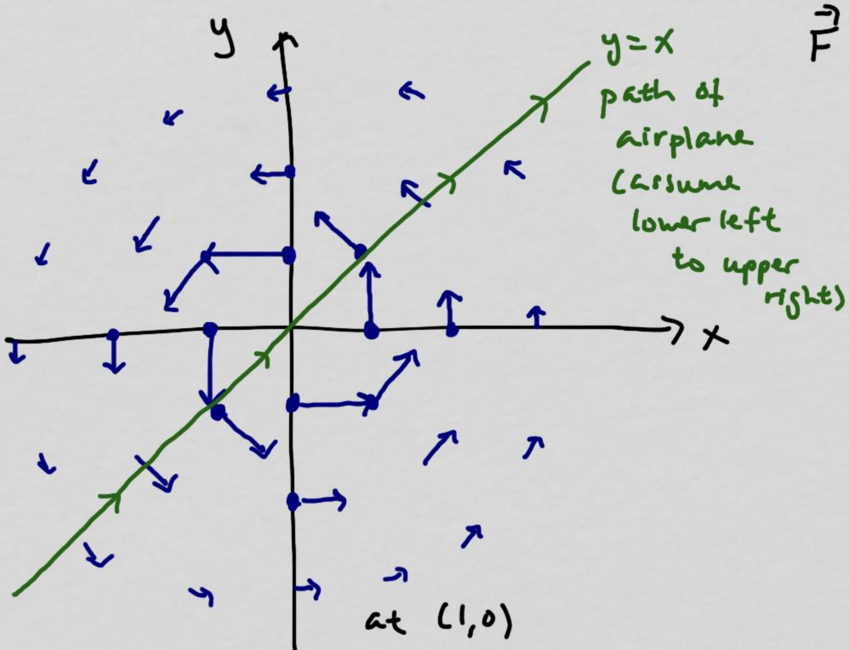
example A hurricane can be modeled by the vector field

$\vec{F} = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$. If an airplane is flying through

the hurricane along the path $y = x$, at what points

are the vector field and the path orthogonal to each other?

at what points are they tangent to each other?



at $(1,0)$

$\langle -y, x\rangle = \langle 0, 1\rangle$

$$\vec{F} = \underbrace{\frac{1}{x^2+y^2}}_{\substack{\text{square}\\ \text{of magnitude of } \langle -y,x\rangle}}\underbrace{\langle -y, x\rangle}_{\text{magnitude } \sqrt{x^2+y^2}}$$

so, each vector is in the direction of $\langle -y, x\rangle$ with magnitude

of $\dfrac{1}{\sqrt{x^2+y^2}} = \dfrac{1}{r}$ → distance from origin

far out → short vectors

close in → long vectors

path of airplane is $y = x$ which can be parametrized as

$$\vec{r}(t) = \langle \underbrace{t, t}_{\text{because } y = x} \rangle \qquad -\infty < t < \infty$$

then at the points where the plane's path intersects the vector field,

the vector field is $\vec{F} = \dfrac{1}{x^2+y^2}\langle -y, x \rangle$ is equal to

$$\vec{F} = \frac{1}{t^2+t^2}\langle -t, t \rangle = \frac{1}{2t^2}\langle -t, t \rangle$$

orthogonal when $\vec{F} \cdot \vec{r} = 0$ $\rightarrow$ $\dfrac{1}{2t^2}\langle -t, t \rangle \cdot \langle \overset{\langle 1, 1 \rangle}{\cancel{t, t}} \rangle = \dfrac{1}{2t^2}(\overset{-t+t}{\cancel{-t^2+t^2}}) = 0$

should be

$\vec{F} \cdot \vec{r}\,' = 0$

because we want the tangent vector of the path to be orthogonal.

so, the dot product is 0 all the time,

$\boxed{\text{the path is always orthogonal to the vectors}}$

tangential if $\vec{F}$ = some multiple of $\vec{r}$

My wrong way works here because for a line through the origin, $\vec{r}$ and $\vec{r}\,'$ are in the same direction

$\vec{F} = \dfrac{1}{2t^2}\langle -t, t \rangle$ $\quad \cancel{\vec{r} = \langle t, t \rangle}$ $\quad \vec{r}\,' = \langle 1, 1 \rangle$ have no chance to be multiples of each other,

$\boxed{\text{the path is never tangential to the hurricane}}$

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if a vector field is the gradient of some scalar field, then the vector field is called a gradient vector field and the scalar field (whose gradient is the vector field) is called the potential function of the gradient vector field

for example, the vector field $\vec{F} = \left\langle \frac{-x}{\left(\sqrt{x^2+y^2}\right)^3}, \frac{-y}{\left(\sqrt{x^2+y^2}\right)^3} \right\rangle$ is the gradient vector field of $U = \frac{1}{\sqrt{x^2+y^2}}$ because

$$\vec{\nabla} U = \vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2}}\right) = \left\langle -\frac{1}{2}(x^2+y^2)^{-3/2}(2x), -\frac{1}{2}(x^2+y^2)^{-3/2}(2y) \right\rangle$$

$$= \left\langle \frac{-x}{\left(\sqrt{x^2+y^2}\right)^3}, \frac{-y}{\left(\sqrt{x^2+y^2}\right)^3} \right\rangle = \vec{F}$$

Since $\vec{F} = \vec{\nabla} U$, we say U is the potential function of $\vec{F}$

and if $\vec{F}$ represents a force, then we say the force is conservative

(because it is the gradient of some potential function)