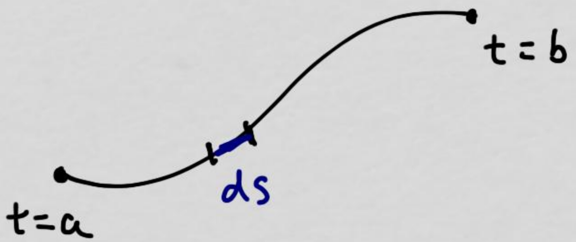
# 17.2 (part 1) Line Integrals of Functions

recall the length of $\vec{r}(t)$, $a \le t \le b$, is

$$L = \int_a^b \underbrace{|\vec{r}\,'|\,dt}_{ds} = \int_C ds$$

$C$ ↳ some curve parametrized by $\vec{r}(t)$




if we integrate some quantity $f(x,y)$ along the curve, then we get

$$\int_C f(x,y)\,ds$$ and this is called a line integral

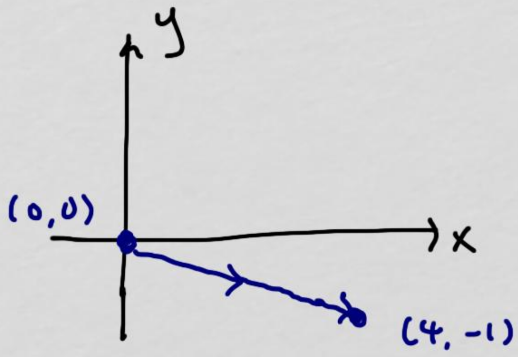
in 3D $\int_C f(x,y,z)\,ds$

the line integral $\int_C f\,ds$ can represent many things. If $f=1$, then $\int_C f\,ds = \int_C ds$ is the length. If $f$ is the density of the material and $C$ is a made made of that material, then $\int_C f\,ds$ gives us the mass of the wire.

basic procedure: parametrize the curve $C$, rewrite $f$ in terms of the parameter, then integrate it along $ds = |\vec{r}\,'|\,dt$

<u>example</u>

$$\int_C x e^{y^2}\, ds \qquad C: \text{line segment from } (0,0) \text{ to } (4,-1)$$




$\int_C x e^{y^2}\, ds$ accumulates $xe^{y^2}$ as we move along the segment

parametrize C: $\vec{r}(t) = \vec{r_0} + t\,\vec{v}$

$\vec{r_0}$ : starting point

$\vec{v}$ : direction vector

here, $\vec{r_0} = \langle 0, 0 \rangle$ and $\vec{v} = \langle 4-0, -1-0 \rangle = \langle 4, -1 \rangle$

$$\vec{r}(t) = \langle 0, 0 \rangle + t \langle 4, -1 \rangle = \langle 4t, -t \rangle \qquad 0 \le t \le 1$$

rewrite the integrand in terms of $t$:

$$x e^{y^2} = \underbrace{(4t)}_{x \text{ of } \vec{r}}\, e^{\overbrace{(-t)}^{y \text{ of } \vec{r}}\,^2} = 4te^{t^2}$$

$$ds = |\vec{r}\,'|\,dt = |\langle 4, -1\rangle|\,dt = \sqrt{17}\,dt$$

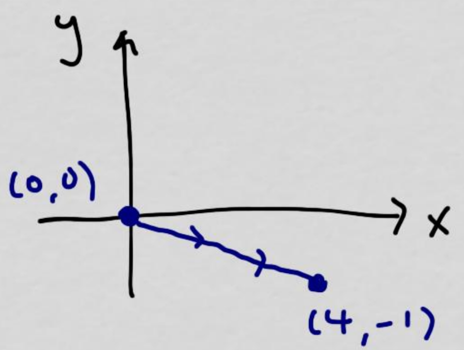
$$\int_C x e^{y^2}\,ds = \int_0^1 4t e^{t^2}\sqrt{17}\,dt = 4\sqrt{17}\underbrace{\int_0^1 t e^{t^2}\,dt}_{\text{by subs}} = \dots = \boxed{2\sqrt{17}\,(e-1)}$$

by subs

$u = t^2 \quad du = 2t\,dt$, etc. .

if $xe^{y^2}$ is the density, then $\int_C xe^{y^2}\,ds$ gives us the mass of the wire in the shape of $C$ which in this case is $2\sqrt{17}\,(e-1)$

what if we had parametrized $C$ differently?




we used $\vec{r}(t) = \langle 4t, -t\rangle \quad 0 \le t \le 1$ earlier

but that is not the only way to parametrize

we could have used $\vec{r}(t) = \langle 8t, -2t\rangle \quad 0 \le t \le 1/2$

would that change the answer?

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we don't expect it to change.

why? If $\int_C f\,ds$ is the mass w/ density $f$, then the mass has no reason to change if we choose to describe the wire differently.

verify: $\int_C x e^{y^2}\,ds$

$C: \vec{r}(t) = \langle 8t, -2t \rangle \quad 0 \le t \le 1/2$

$\vec{r}\,' = \langle 8, -2 \rangle$

$|\vec{r}\,'| = \sqrt{68} \qquad ds = |\vec{r}\,'|\,dt = \sqrt{68}\,dt$

$$= \int_0^{1/2} (8t) e^{(-2t)^2} \sqrt{68}\,dt = 8\sqrt{68} \int_0^{1/2} t e^{4t^2}\,dt$$

$u = 4t^2$
$du = 8t\,dt$

$u = 4(1/2)^2$

$$= 8\sqrt{68} \int_0^1 \frac{1}{8} e^u\,du = \sqrt{68}\int_0^1 e^u\,du = \sqrt{68}\,(e-1) = \sqrt{4\cdot 17}\,(e-1)$$

$u = 4(0)^2$

$\sqrt{68} \to 4\cdot 17$

$$= \boxed{2\sqrt{17}\,(e-1)}$$

Same result

$$\boxed{\int_C f\,ds \text{ does } \underline{\text{NOT}} \text{ depend on the choice of parametrization}}$$

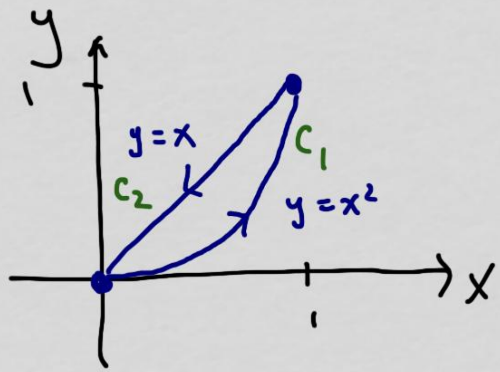
So, choose the simplest or the most convenient parametrization

<u>example</u>

$$\int_C (x+\sqrt{y})\,ds$$

C : from $(0,0)$ to $(1,1)$ along $y=x^2$

then back to $(0,0)$ along $y=x$




parametrize : $C_1$ : $\vec{r}(t) = \langle \underbrace{t,}\, t^2 \rangle \quad 0 \le t \le 1$

because $y = x^2$

$C_2$ : $\vec{r}(t) = \langle 1-t, 1-t \rangle \quad 0 \le t \le 1$

because

$x = y$

remember, we can parametrize however we want

$C_1 : \vec{r}\,' = \langle 1, 2t \rangle, \quad |\vec{r}\,'| = \sqrt{1+4t^2} \quad \text{so} \quad ds = \sqrt{1+4t^2}\,dt \quad 0 \le t \le 1$

$C_2 : \vec{r}\,' = \langle -1, -1 \rangle \quad |\vec{r}\,'| = \sqrt{2} \quad \text{so} \quad ds = \sqrt{2}\,dt \quad 0 \le t \le 1$

$$\int_C (x + \sqrt{y})\,ds$$

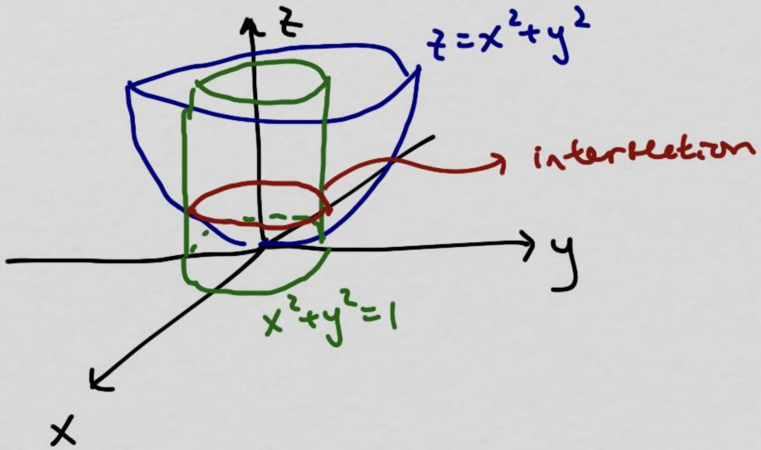
$$= \underbrace{\int_0^1 \Big(\underset{\substack{\uparrow \\ x \text{ on } C_1}}{t} + \underset{\substack{\uparrow \\ y \text{ on } C_1}}{\sqrt{t^2}}\Big) \underbrace{\sqrt{1+4t^2}\,dt}_{ds \text{ on } C_1}}_{C_1} + \underbrace{\int_0^1 \Big(\underbrace{1-t}_{x \text{ on } C_2} + \sqrt{\underbrace{1-t}_{y \text{ on } C_2}}\Big) \underbrace{\sqrt{2}\,dt}_{ds \text{ on } C_2}}_{C_2}$$

$$= \cdots = \boxed{\frac{1}{6}(5\sqrt{5} - 1) + \frac{7}{3\sqrt{2}}}$$

**example**

$$\int_C (x+y+z)\, ds$$

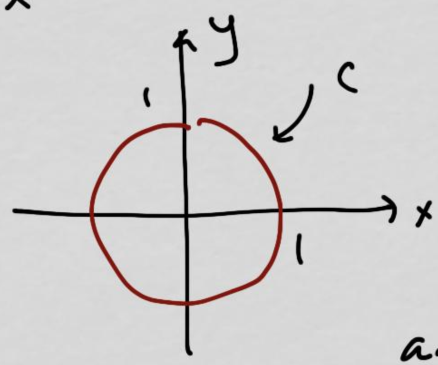
$C$ : intersection of the paraboloid $z = x^2+y^2$ and the cylinder $x^2+y^2=1$




$z = x^2+y^2$ , $\underbrace{x^2+y^2=1}$

$z=1$ intersect at $z=1$

$x^2+y^2=1$ has a circle of radius at all $z$ as its contour, including $z=1$




$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle \quad 0 \le t \le 2\pi$

at $z=1$

another possible choice : $\vec{r}(t) = \langle \cos 2t, \sin 2t, 1 \rangle \quad 0 \le t \le \pi$

$$\int_C (x+y+z)\,ds$$

$C: \quad \vec{r}(t) = \langle \cos t, \sin t, 1 \rangle \quad 0 \le t \le 2\pi$

$\vec{r}\,' = \langle -\sin t, \cos t, 0 \rangle$

$|\vec{r}\,'| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (0)^2} = 1$

So, $ds = |\vec{r}\,'|\,dt = dt$

$$= \int_0^{2\pi} \Big(\underset{x \text{ of } \vec{r}}{\cos t} + \underset{y \text{ of } \vec{r}}{\sin t} + \underset{z \text{ of } \vec{r}}{1}\Big)\,\overset{ds}{dt} = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$$