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17.3 Conservative Vector Fields and the Fundamental Theorem of Line Integrals

a vector field $\vec{F}$ is conservative if it is the gradient of some potential function $\vec{F} = \vec{\nabla}\phi$

if given ϕ, $\vec{F}$ is to find

but if given $\vec{F}$, how do we know if $\vec{F}$ is conservative?

and if $\vec{F}$ is conservative, how to find ϕ?

let $\vec{F} = \langle f, g \rangle$ be a conservative vector field

then we know $\vec{F} = \langle f, g \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$

so $f = \frac{\partial \phi}{\partial x}$

$g = \frac{\partial \phi}{\partial y}$

we also know that $\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial \phi}{\partial x}}_{f}\Big) = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial \phi}{\partial y}}_{g}\Big) \longrightarrow \phi_{yx} = \phi_{xy}$

$$\boxed{\text{So, this means, if } \vec{F} = \langle f, g \rangle \text{ is conservative, then } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad f_y = g_x}$$

<u>example</u> $\vec{F} = \langle \underset{f}{x}, \underset{g}{y} \rangle$

is $f_y = g_x$? $\quad f_y = \frac{\partial}{\partial y}(x) = 0 \quad g_x = \frac{\partial}{\partial x}(y) = 0$

yes, so $\vec{F} = \langle x, y \rangle$ <u>is conservative</u>.

<u>example</u> $\vec{F} = \langle \underset{f}{-y}, \underset{g}{x} \rangle$

is $f_y = g_x$? $\quad f_y = -1, \quad g_x = 1$

no, so $\vec{F}$ is <u>NOT</u> conservative example $\vec{F} = \langle x+y, x \rangle$

f g

is $f_y = g_x$? $f_y = 1$, $g_x = 1$

yes, so $\vec{F}$ is conservative.

If we know $\vec{F} = \langle f, g \rangle$ is conservative $\rightarrow \vec{F} = \vec{\nabla} \phi$

how do we find ϕ?

let's use $\vec{F} = \langle x+y, x \rangle$ as an example

$$\vec{F} = \langle x+y, x \rangle = \vec{\nabla} \phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$$

so, $$\frac{\partial \phi}{\partial x} = x + y \quad (1)$$

$$\frac{\partial \phi}{\partial y} = x \quad (2)$$

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(1) : $\dfrac{\partial \phi}{\partial x} = x + y$

integrate with respect to x

$$\phi = \int (x + y)\,d\underset{=}{x} = \frac{1}{2}x^2 + xy + \underbrace{a(y)} \quad (3)$$

variable

treat as constant

some function that can depend on y only (or is a constant)
because if this function were to have x, then it's derivative would show up if $\frac{\partial \phi}{\partial x}$ (1)
because a does not on x, it vanishes in $\frac{\partial \phi}{\partial x}$ (1)

now let's take the partial deriv. of this ϕ with respect to y and compare to (2)

$$\frac{\partial \phi}{\partial y} = x + \frac{da}{dy} = x$$

from (2) $\frac{\partial \phi}{\partial y} = x$

so $\dfrac{da}{dy} = 0 \rightarrow a = \text{Constant} = C$

now sub that into (3)

$$\boxed{\phi(x, y) = \frac{1}{2}x^2 + xy + C}$$

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check: is $\vec{\nabla}\phi = \vec{F} = \langle x+y, x \rangle$?

$$\vec{\nabla}\left(\tfrac{1}{2}x^2 + xy + c\right) = \langle x+y, x \rangle$$ so, yes, therefore ϕ is correct.

what about 3D $\vec{F} = \langle f, g, h \rangle$?

how to check if it is conservative?

and if so, how to find ϕ?

let $\vec{F} = \langle f, g, h \rangle$ be a conservative vector field

then $\vec{F} = \langle f, g, h \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$

so, $f = \frac{\partial \phi}{\partial x}$ we know $\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) \longrightarrow f_y = g_x$

$g = \frac{\partial \phi}{\partial y}$ $\qquad \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) \longrightarrow g_z = h_y$

$h = \frac{\partial \phi}{\partial z}$ $\qquad \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial z}\right) \longrightarrow f_z = h_x$

So, if $\vec{F} = \langle f, g, h \rangle$ is conservative,

then
$$\left.\begin{aligned} f_y &= g_x \\ f_z &= h_x \\ g_z &= h_y \end{aligned}\right\} \text{ all three must be satisfied}$$

example $\vec{F} = \langle \underset{f}{x^2 - ze^y}, \underset{g}{y^3 - xze^y}, \underset{h}{z^4 - xe^y} \rangle$

is $f_y = g_x$? $f_y = -ze^y$, $g_x = -ze^y$ yes

is $f_z = h_x$? $f_z = -e^y$, $h_x = -e^y$ yes

is $g_z = h_y$? $g_z = -xe^y$, $h_y = -xe^y$ yes

so, $\vec{F}$ is conservative and $\vec{F} = \vec{\nabla}\phi$

we find ϕ using a very similar process used in last example

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$$\vec{F} = \langle x^2 - ze^y,\ y^3 - xze^y,\ z^4 - xe^y \rangle = \vec{\nabla}\phi = \langle \phi_x, \phi_y, \phi_z \rangle$$

$$\phi_x = x^2 - ze^y \qquad (1)$$

$$\phi_y = y^3 - xze^y \qquad (2)$$

$$\phi_z = z^4 - xe^y \qquad (3)$$

integrate (1) with respect to $x \rightarrow \phi = \int (x^2 - ze^y)\,dx$

function of y and z or a constant (deriv. with x is 0 so is not in ϕ_x)

$$\phi = \frac{1}{3}x^3 - xze^y + a(y,z) \qquad (4)$$

differentiate (4) with respect to y and compare to (2)

$$\phi_y = -xze^y + \frac{\partial a}{\partial y} = \underbrace{y^3 - xze^y}_{\text{from } (2)} \longrightarrow \frac{\partial a}{\partial y} = y^3$$

differentiate (4) with respect to z and compare to (3)

$$\phi_z = -xe^y + \frac{\partial a}{\partial z} = \underbrace{z^4 - xe^y}_{\text{from } (4)} \longrightarrow \frac{\partial a}{\partial z} = z^4$$

now we find $a(y,z)$ from $\underbrace{\frac{\partial a}{\partial y} = y^3}$ and $\frac{\partial a}{\partial z} = z^4$

$$a = \int y^3 \, dy$$

$$a = \frac{1}{4} y^4 + b(z)$$

some function of z

differentiate with respect to z and compare to $\frac{\partial a}{\partial z}$

$$\frac{\partial a}{\partial z} = \frac{db}{dz} = z^4 \longrightarrow b = \frac{1}{5} z^5 + C$$

↳ constant

so, $a = \frac{1}{4} y^4 + \frac{1}{5} z^5 + C$ sub this into ϕ (4)

and we get

$$\boxed{\phi(x,y,z) = -\frac{1}{3} x^3 - xze^y + \frac{1}{4} y^4 + \frac{1}{5} z^5 + C}$$

if unsure, find its gradient and check if $\vec{\nabla} \phi = \vec{F}$

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why do we care if $\vec{F}$ is conservative and why do we care about ϕ?

because if $\vec{F}$ is conservative ($\vec{F} = \vec{\nabla}\phi$)

then $\int_C \vec{F}\cdot\vec{T}\,ds$ is independent of the path C

(the only things that matter are the starting and ending locations of C)

why? $\int_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\int_C \vec{F}\cdot\vec{r}\,'\,dt} = \int_C \vec{F}\cdot d\vec{r}$

let $\vec{F} = \langle f, g\rangle$ and $\vec{r}(t) = \langle x, y\rangle$ $a \le t \le b$

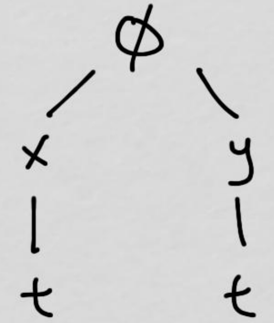
if $\vec{F} = \vec{\nabla}\phi$

then $\int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{\nabla}\phi\cdot\underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle}_{\vec{r}\,'}\,dt = \int_C \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right\rangle\cdot\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$

$$= \int_C \underbrace{\left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \right)}_{\frac{d\phi}{dt}} dt$$

why?

$$\phi(x,y) = \phi\left(\underbrace{x(t),\ y(t)}_{\text{from parametritation of } \vec{r}(t)}\right)$$



ϕ is ultimately a function of t

step down tree : $\dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial \phi}{\partial y}\dfrac{dy}{dt}$

$$= \int_C \frac{d\phi}{dt}\, dt$$

$$= \int_C d\phi$$

$$= \phi(B) - \phi(A)$$

end location of path C

~~end~~ start location of path C

notice this says nothing about how we actually go from A to B

this result is called the **Fundamental Theorem of Line Integrals**

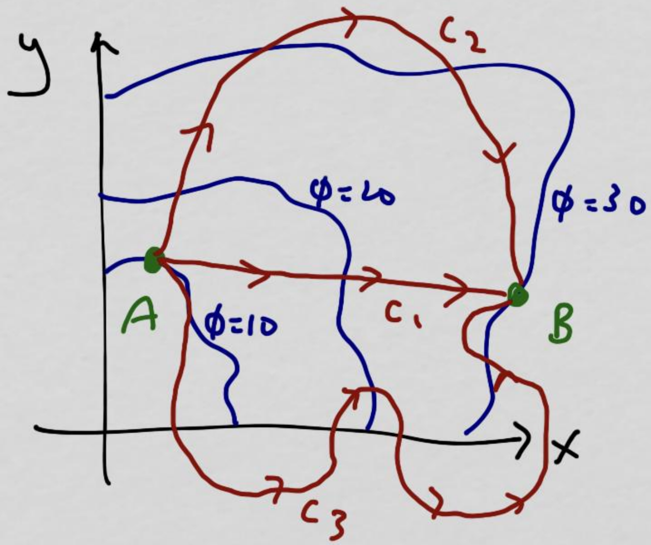
if $\vec{F} = \vec{\nabla}\phi$

then $\int_c \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$

regardless of the actual path c

B: ending location A: starting location

graphically, this means that only the values of ϕ at start and end matter



ALL three paths shown give the same

$$\int_c \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$$

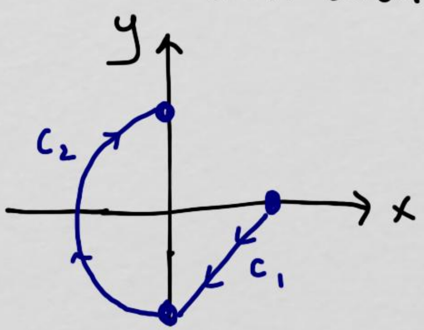
$$= 30 - 10 = 20$$

path is irrelevant if $\vec{F}$ is conservative

example $\int_C \vec{F} \cdot d\vec{r} \qquad \vec{F} = \langle x+y,\ x \rangle$

C: line segment from $(1,0)$ to $(0,-1)$

then along the left half of $x^2+y^2=1$ to $(0,1)$



let's first do this as a regular line integral

$C_1: \ \vec{r}_1(t) = \langle 1-t, -t \rangle \qquad 0 \le t \le 1$

$C_2: \ \vec{r}_2(t) = \langle -\sin t, -\cos t \rangle \qquad 0 \le t \le \pi$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \underbrace{\langle 1-2t,\ 1-t \rangle}_{\substack{\vec{F} \text{ using} \\ x,y \text{ of } \vec{r}_1}} \cdot \underbrace{\langle -1, -1 \rangle}_{\vec{r}_1\,'}\, dt + \int_0^\pi \underbrace{\langle -\sin t - \cos t,\ -\sin t \rangle}_{\substack{\vec{F} \text{ w/ } x,y \\ \text{of } \vec{r}_2}} \cdot \underbrace{\langle -\cos t, \sin t \rangle}_{\vec{r}_2\,'} dt$$

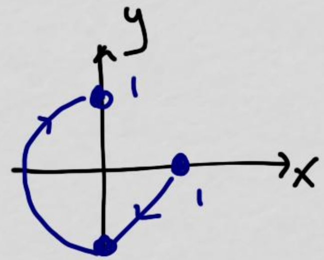
$$= \int_0^1 (3t-2)\,dt + \int_0^\pi \Big(\underbrace{\sin t \cos t}_{\substack{\text{subs.} \\ u = \sin t}} + \underbrace{\cos^2 t}_{\frac{1+\cos 2t}{2}} - \underbrace{\sin^2 t}_{\frac{1-\cos 2t}{2}}\Big)\, dt = \dots = \boxed{-\frac{1}{2}}$$

using the Fundamental Theorem of Line Integrals:

first, find ϕ such that $\vec{\nabla}\phi = \vec{F} = \langle x+y, x \rangle$

from earlier example, $\phi = \frac{1}{2}x^2 + xy + C$

then $\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$



$(0, 1)$ $\quad$ $(1, 0)$

$x \;\; y$ $\quad$ $x \;\; y$

$$= \left[\frac{1}{2}(0)^2 + (0)(1) + C\right] - \left[\frac{1}{2}(1)^2 + (1)(0) + C\right] = \boxed{-\frac{1}{2}}$$

since $\vec{F}$ is conservative, we could also pick an easier path

if we wanted evaluate $\int_C \vec{F} \cdot d\vec{r}$ as a regular line integral