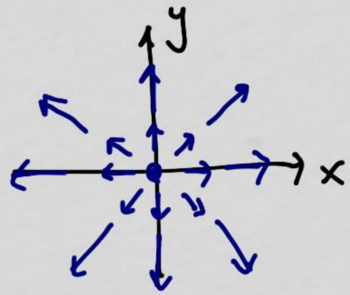
17.4 Green's Theorem

for a 2D vector field $\vec{F} = \langle f, g \rangle$ the vector quantity $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$ or $\langle 0, 0, g_x - f_y \rangle$ is called the <u>curl</u> of $\vec{F}$ written as <u>curl $\vec{F}$</u>.

$|\text{curl}\,\vec{F}| = g_x - f_y$, a scalar quantity is a measure of the amount of rotation in the vector field.

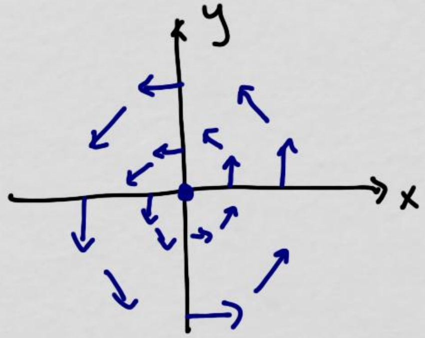
for example, $\vec{F} = \langle x, y \rangle$



clearly has no rotation

its curl is $\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle = \langle 0, 0, 0 - 0 \rangle = \vec{0}$

and $|\text{curl}\,\vec{F}| = 0 \rightarrow$ indicating no rotation (which agrees with the conclusion from visual inspection) in contrast, $\vec{F} = \langle -y, x \rangle$



clearly has rotation

curl $\vec{F} = \langle 0, 0, g_x - f_y \rangle = \langle 0, 0, 1-(-1) \rangle = \langle 0, 0, 2 \rangle$

$|\text{curl}\,\vec{F}| = 2 \rightarrow$ not zero, indicating that the vector field is rotating

last time, we saw that if $\vec{F} = \langle f, g \rangle$ is conservative, then $\underbrace{g_x - f_y}_{\text{magnitude of curl }\vec{F}} = 0$

this means that <u>a conservative vector field has no rotation</u> $\rightarrow$ the vector field is <u>irrotational</u>

now we can look at and understand <u>Green's Theorem</u>

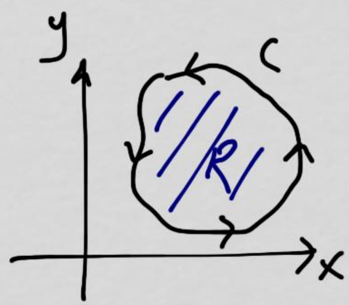
Green's Theorem

if $\vec{F} = \langle f, g \rangle$ is a vector field and C is simple closed path in a plane traversed in the counterclockwise direction, then

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

means C is a closed loop (the circle on the integral sign is optional)

the region enclosed by C

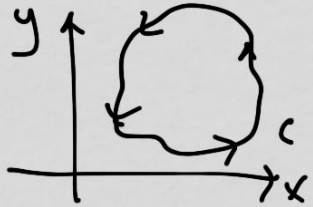


why is this true?

how can we say a line integral is somehow equal to a double integral over a region?

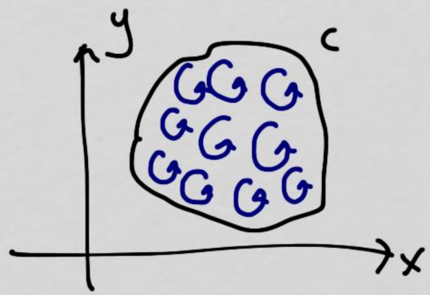
left side: $\displaystyle\int_C f\,dx + g\,dy = \int_C \underbrace{\langle f, g\rangle}_{\vec F} \cdot \underbrace{\langle dx, dy\rangle}_{d\vec r \text{ with } \vec r(t) = \langle x, y\rangle}$

this is a line integral over a closed path → <u>circulation</u> around the boundary of R

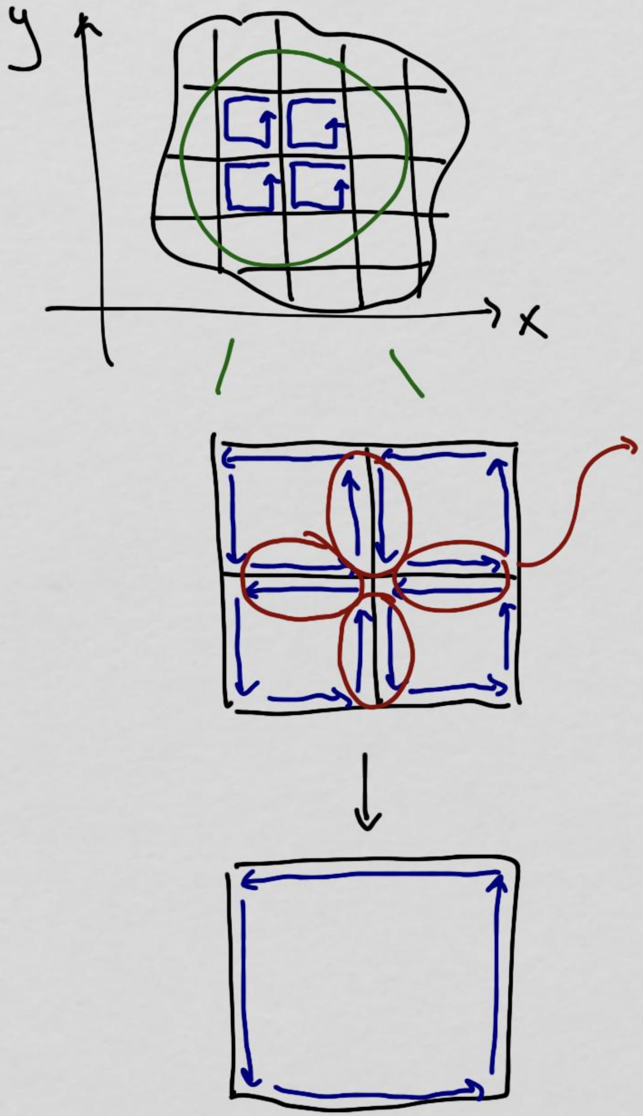


right side: $\displaystyle\iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)}_{\text{part of curl (measure of rotation)}} dA$

this integral accumulates the tiny rotations in the field throughout R

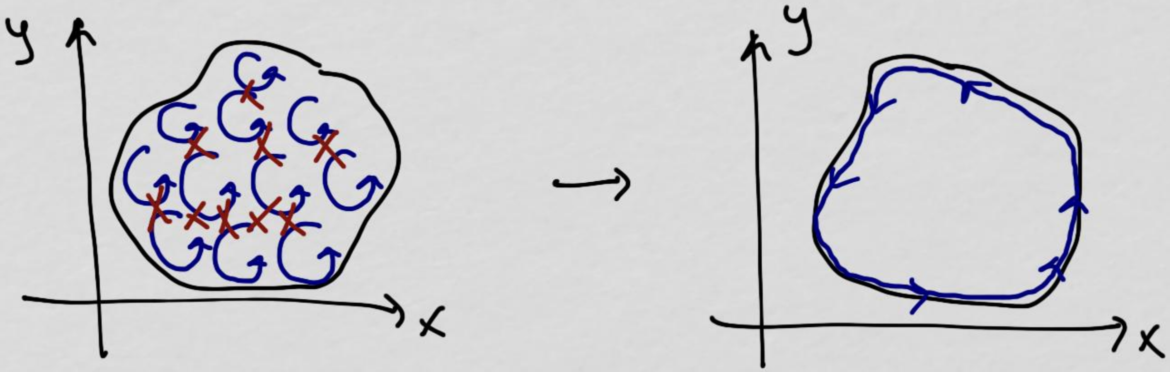


the rotations partially cancel one another out



equal and opposite flows
at shared boundaries
so the flows cancel out there

so, all the interior flows and rotations eventually vanish, and only the flow on the boundary of R remains



which is precisely what the left side ($\oint_C f\,dx + g\,dy$) of the theorem is calculating. This is why Green's Theorem is true.

Green's Theorem allows us to trade a line integral for a double integral (or vice versa)

$$\oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$$

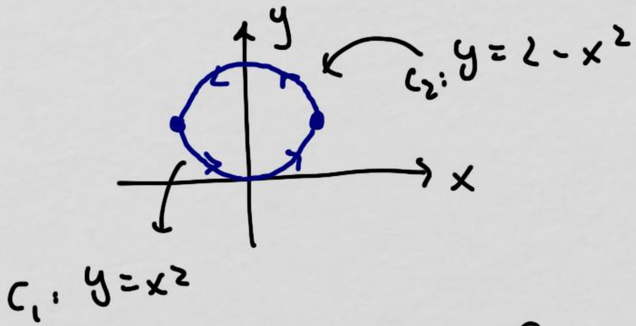
if $\vec{F}$ is conservative, then $g_x - f_y = 0$, so Green's Theorem says

$$\oint_C f\,dx + g\,dy = \oint_C \vec{F}\cdot d\vec{r} = \underbrace{\oint_C \vec{\nabla}\phi \cdot d\vec{r} = 0}$$

agrees with the Fundamental Theorem of Line Integrals $\oint_C \vec{\nabla}\phi\, d\vec{r} = \phi(B) - \phi(A) = 0$

<u>example</u> $\vec{F} = \langle y+2,\ x^2+1 \rangle$

C: $(-1,1)$ to $(1,1)$ along $y = x^2$ then back to $(-1,1)$ along $y = 2-x^2$



Green's Theorem requires the direction to be counterclockwise. If not, flip the sign on one side

let's calculate $\oint_C \vec{F}\cdot d\vec{r}$ in two ways: as line integral
then using Green's Theorem

<u>line integral</u> C_1: $\vec{r}_1 = \langle t, t^2 \rangle \quad -1 \le t \le 1$

C_2: $\vec{r}_2 = \langle -t, 2-t^2 \rangle \quad -1 \le t \le 1$

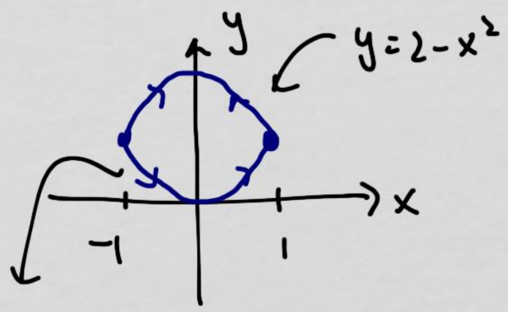
$$\oint_C f\,dx + g\,dy = \oint_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \underbrace{\langle t^2+2,\ t^2+1 \rangle}_{\substack{\vec{F} \text{ using } x,y \\ \text{on } \vec{r}_1}} \cdot \underbrace{\langle dt,\ 2t\,dt \rangle}_{d\vec{r}_1} + \int_{-1}^{1} \langle 4-t^2,\ t^2+1 \rangle \cdot \langle -dt, -2t\,dt \rangle$$

$$= \int_{-1}^{1} (2t^3+t^2+2t+2)\,dt + \int_{-1}^{1} (-2t^3+t^2-2t-4)\,dt = \cdots = \frac{14}{3} - \frac{22}{3} = \boxed{-\frac{8}{3}}$$

using Green's Theorem :

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (g_x - f_y)\,dA$$

$$\vec{F} = \langle y+2,\ x^2+1\rangle \qquad g_x - f_y = 2x-1$$



y=x²

$$R:\ -1 \le x \le 1$$
$$x^2 \le y \le 2-x^2$$

$$\iint_R (g_x - f_y)\,dA = \int_{-1}^{1}\int_{x^2}^{2-x^2} (2x-1)\,dy\,dx = \int_{-1}^{1} 2xy - y\Big|_{y=x^2}^{y=2-x^2} dx$$

$$= \int_{-1}^{1} (4x-2x^3-2+x^2-2x^3+x^2)\,dx = \int_{-1}^{1}(-4x^3+2x^2+4x-2)\,dx = \boxed{-\frac{8}{3}}$$

one way is not necessarily easier than the other

but the theorem gives us an option

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Green's Theorem also let's us calculate the area of the region enclosed by C by using a line integral

$$\oint_C f\,dx + g\,dy = \iint_R \underbrace{(g_x - f_y)}_{}\,dA$$

if this is 1, then right side is $\iint_R dA$ = area of R = area of region enclosed by C

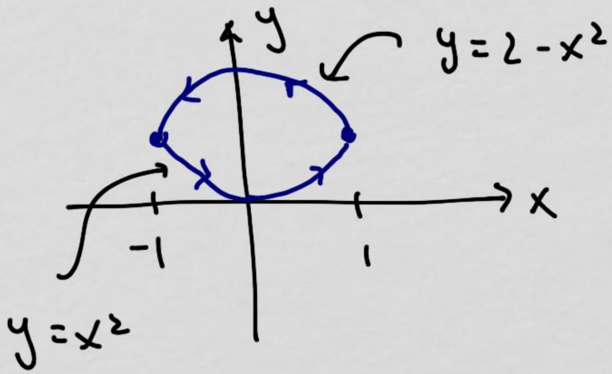
So, we need to find an $\vec{F} = \langle f, g \rangle$ such that $g_x - f_y = 1$ and then use those f and g on the left side line integral to find area.

note there are infinitely-many $\vec{F} = \langle f, g \rangle$ such that $g_x - f_y = 1$

here is one: $\vec{F} = \langle -\frac{1}{2}y, \frac{1}{2}x \rangle$

so, left side is $\oint_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy = \boxed{\frac{1}{2}\oint_C x\,dy - y\,dx}$ = area enclosed by C

<u>example</u> Area enclosed by C in the previous example



$C_1: \vec{r}_1 = \langle t, t^2 \rangle \quad -1 \le t \le 1$

$C_2: \vec{r}_2 = \langle -t, 2-t^2 \rangle \quad -1 \le t \le 1$

area enclosed is $\frac{1}{2} \oint_C x\,dy - y\,dx$

$$= \underbrace{\frac{1}{2}\int_{-1}^{1} \underbrace{(t)}_{x}\underbrace{(2t\,dt)}_{dy} - \underbrace{(t^2)}_{y}\underbrace{(dt)}_{dx}}_{C_1} + \underbrace{\frac{1}{2}\int_{-1}^{1} \underbrace{(-t)}_{x}\underbrace{(-2t\,dt)}_{dy} - \underbrace{(2-t^2)}_{y}\underbrace{(-dt)}_{dx}}_{C_2}$$

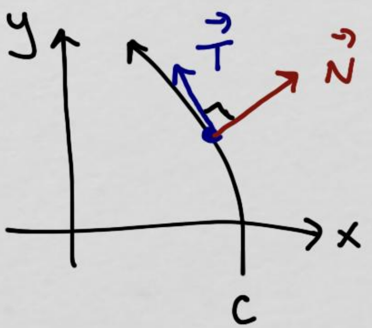
$$= \frac{1}{2}\int_{-1}^{1} t^2\,dt + \frac{1}{2}\int_{-1}^{1} (t^2+2)\,dt = \frac{1}{3} + \frac{7}{3} = \boxed{\frac{8}{3}}$$

We can also use Green's Theorem to calculate the flux integral $\int_C \vec{F} \cdot \vec{N}\, ds$

unit normal vector to the right of unit tangent

let $\vec{F} = \langle f, g \rangle$

$\vec{r}(t) = \langle x, y \rangle$



since $\vec{N}$ is to the right of $\vec{T}$ we see that $\vec{N} = \vec{T} \times \vec{k}$

↳ unit vector in z direction (out of page)

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{\langle x', y' \rangle}{\sqrt{(x')^2 + (y')^2}}$$

$$\text{so } \vec{N} = \vec{T} \times \vec{k} = \frac{1}{\sqrt{(x')^2 + (y')^2}} \langle x', y', 0 \rangle \times \underbrace{\langle 0, 0, 1 \rangle}_{\vec{k}}$$

$$\vec{N} = \frac{\langle y', -x' \rangle}{\sqrt{(x')^2 + (y')^2}}$$

$$\oint_C \vec{F}\cdot\vec{N}\,ds = \oint_C \langle f, g\rangle \cdot \frac{\langle y', -x'\rangle}{\cancel{\sqrt{(x')^2+(y')^2}}}\,\cancel{\sqrt{(x')^2+(y')^2}}\,dt$$

$$= \oint_C \langle f, g\rangle \cdot \langle y', -x'\rangle\,dt$$

$$= \oint_C f\,\underbrace{y'dt} - g\,\underbrace{x'dt}$$

$\frac{dy}{dt}dt = dy$ $\qquad$ $\frac{dx}{dt}dt = dx$

$$= \oint_C f\,dy - g\,dx$$

turn into a double integral using Green's Theorem

$$\oint_C f\,dx + g\,dy = \iint_R (\underbrace{g_x} - \widetilde{f_y})\,dA$$

→ the part w/ dx is differentiated with respect to y

goes w/ dx

goes w/ dy

the part w/ dy is differentiated with respect to x

so, $\oint_C f\,dy - g\,dx$ becomes

$$= \iint_R \big[\underbrace{f_x}_{\text{the part that went w/ dy}} - \underbrace{(-g)_y}_{\text{the part that went w/ dx}} \big]\, dA = \boxed{\iint_R (f_x + g_y)\, dA}$$

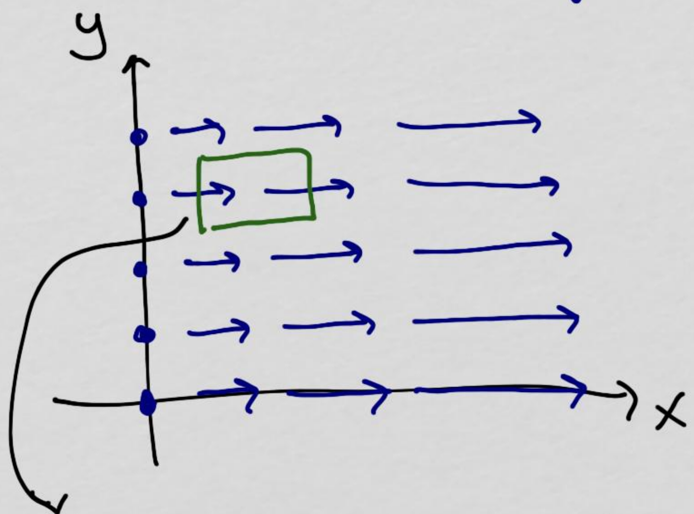
this calculates the flux through C (outward flow through C)

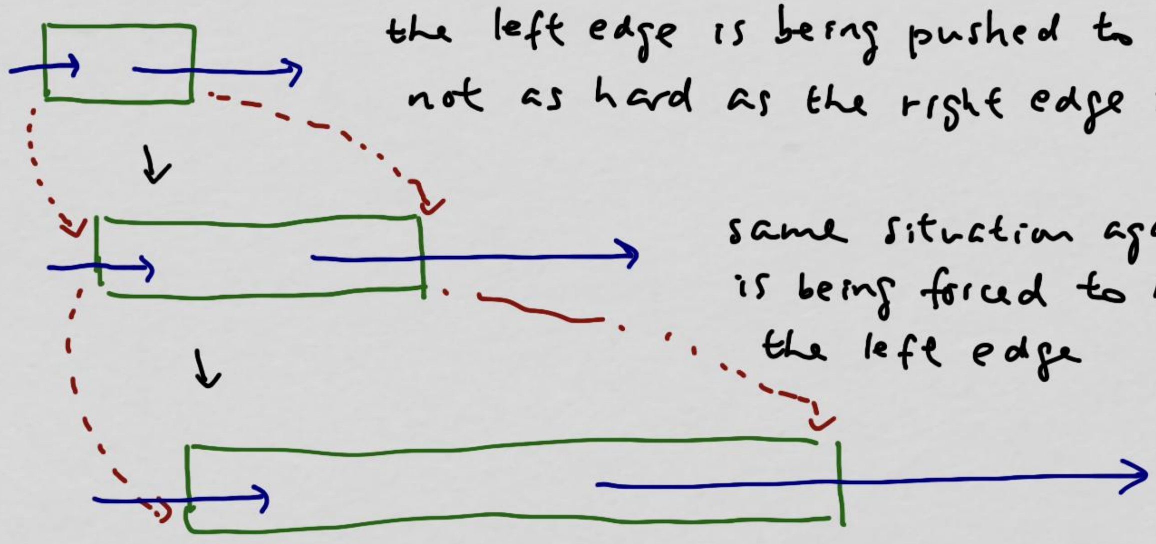
the quantity being accumulated, $f_x + g_y$, is called the

<u>divergence</u> of $\vec{F} = \langle f, g \rangle \to \boxed{\operatorname{div} \vec{F} = f_x + g_x}$

the divergence measures the volume change of a little box that goes with the flow of the vector field for example, $\vec{F} = \langle \underset{f}{x}, \underset{g}{0} \rangle \to \operatorname{div} \vec{F} = f_x + g_y = 1$

indicates that a small box flowing through the vector field increases in volume





the left edge is being pushed to the right but not as hard as the right edge is being pushed

same situation again, the right edge is being forced to move away from the left edge the box stretches and increases in volume as it flows down the vector field with the flow

→ increase in volume (positive divergence)

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