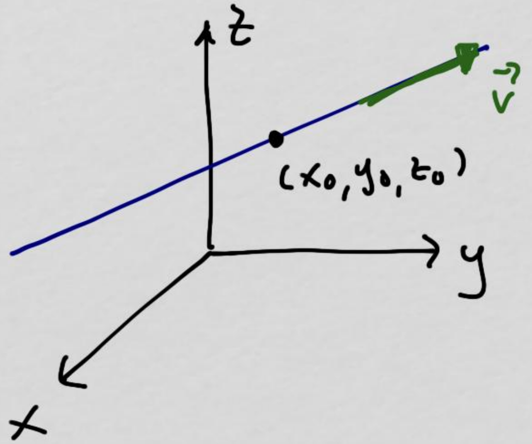
13.5 Lines and Planes in Space

What is a line?

It's a collection of points that all lie along the same direction



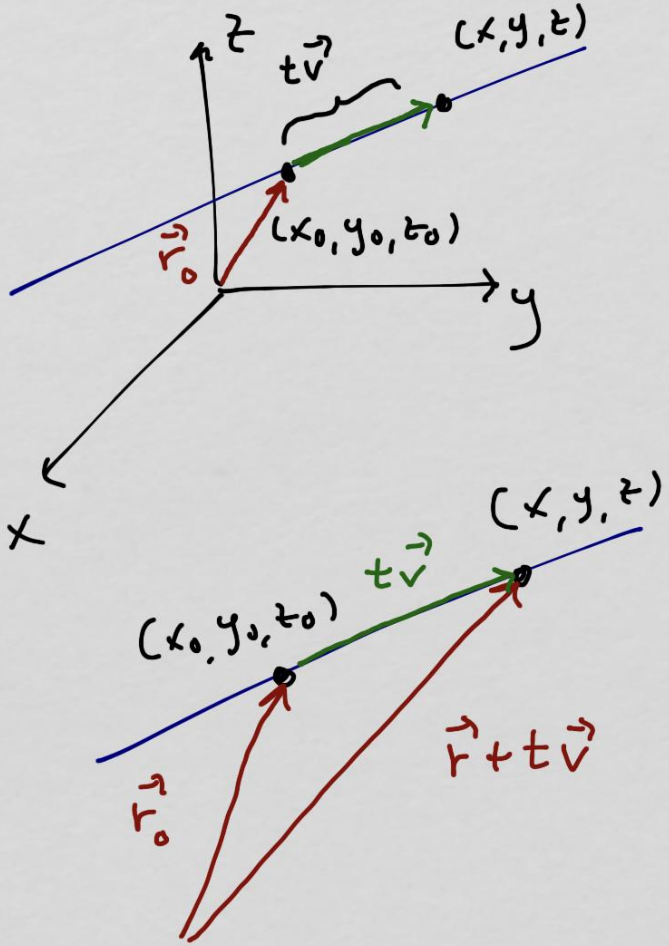
(x_0, y_0, z_0) is a point that the lines goes through

$\vec{v}$: the vector that gives the line a direction called the <u>direction vector</u>

(the vector that relates the location of one point from another)

How do we describe the location of any point on a line?

once we have that, it gives us an equation of the line



$\vec{r}_0$: position vector of the point (x_0, y_0, z_0) from origin

$t\vec{v}$: vector along the direction vector with scaling constant t

to find another point (x,y,z), we add $t\vec{v}$ to $\vec{r}_0$ and by varying t we can reach any and all points on the line

(an thus an equation of the line)

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So, the equation of a line is $\boxed{\vec{r}(t) = \vec{r}_0 + t\vec{v}}$ <u>vector form</u>

$\vec{r}_0$: vector from origin to (x_0, y_0, z_0)

$\vec{v}$: direction vector

if we write out the components explicitly

$\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

$\vec{v} = \langle a, b, c \rangle$

then we get $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

or $\boxed{x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct}$ parametric form

(t is the parameter)

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<u>example</u> : Line through $(1, 2, 3)$ with direction vector $\vec{v} = \langle 4, 5, 6 \rangle$

vector form: $\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$

$= \langle 1, 2, 3 \rangle + t \langle 4, 5, 6 \rangle$

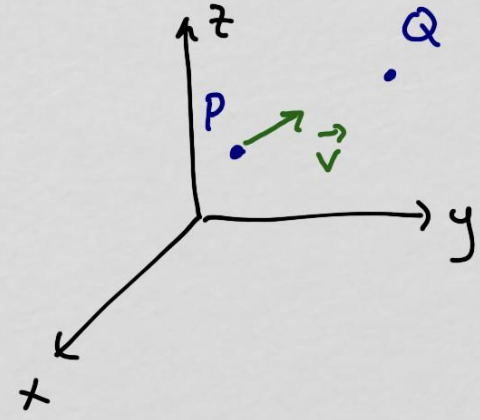
$= \boxed{\langle 1+4t,\ 2+5t,\ 3+6t \rangle}$ vector form

parametric form: $\boxed{x = 1+4t,\quad y = 2+5t,\quad z = 3+6t}$

Since this line is infinitely long, $-\infty < t < \infty$

and $t = 0$ corresponds to the given point

example Line segment joining the points $P(0,1,2)$ $Q(-3,4,7)$



direction vector: any vector in the direction from P to Q

one easy way: $\vec{v} = \vec{PQ} = \langle -3, 3, 5 \rangle$

then let vector from origin to P be $\vec{r}_0$

$$\vec{r}_0 = \langle 0, 1, 2 \rangle$$

vector form: $\vec{r} = \vec{r}_0 + t\vec{v} = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle$

$$= \boxed{\langle -3t, 1+3t, 2+5t \rangle}$$

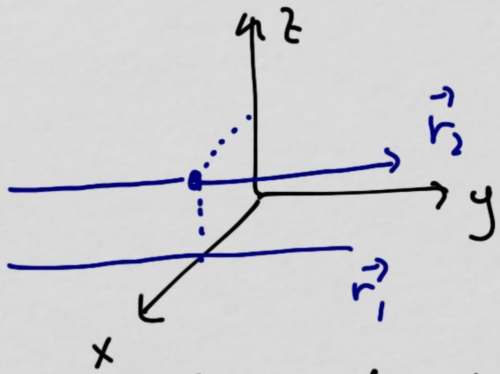
line ends, so we need to know the permissible values of t

$$\left.\begin{array}{l} t=0 \rightarrow \vec{r}(0) = \langle 0, 1, 2 \rangle \text{ at } P \\ t=1 \rightarrow \vec{r}(1) = \langle -3, 4, 7 \rangle \text{ at } Q \end{array}\right\} \text{ so, } \boxed{0 \le t \le 1}$$

two lines are parallel if their direction vectors are multiples of each other

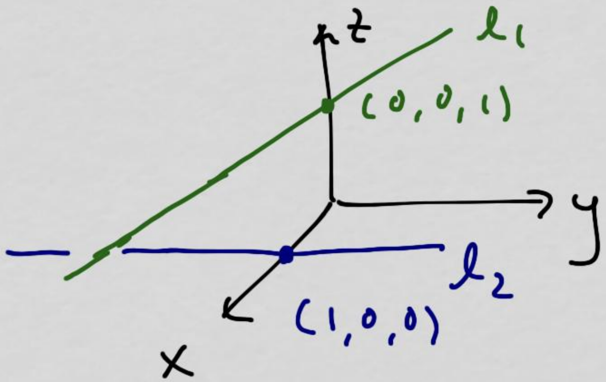
$\vec{r}_1(t) = \langle 1, 0, 0 \rangle + t \langle 0, 1, 0 \rangle$

$\vec{r}_2(t) = \langle 1, 0, 1 \rangle + t \langle 0, 2, 0 \rangle$



the two line could be on top of each other (same line) if they go through a common point

note that in $\mathbb{R}^2$ (2D space) if two lines are not parallel, then they will intersect, but that is not true in $\mathbb{R}^3$



l_1 is parallel to x-axis

l_2 is parallel to y-axis

they are not parallel, but they don't intersect $\rightarrow$ lines are <u>skew</u>

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<u>example</u> Two objects traveling on lines given by

$$\vec{r}_1(t) = \langle 2t+3,\ 4t+2,\ 3t+5 \rangle \qquad -\infty < t < \infty$$

$$\vec{r}_2(s) = \langle s+2,\ 3s-1,\ -5s+10 \rangle \qquad -\infty < s < \infty$$

Will their paths intersect?

Will the two objects collide?

intersect: we can find t and s such that $\vec{r}_1(t) = \vec{r}_2(s)$

collision: we can find t and s such that $\vec{r}_1(t) = \vec{r}_2(s)$
AND $t = s$ (the two objects at at the same location at the same time)

if $\vec{r}_1(t) = \vec{r}_2(s)$ then the components must match

x: $2t+3 = s+2$ — 1

y: $4t+2 = 3s-1$ — 2

z: $3t+5 = -5s+10$ — 3

from ①, $s = 2t + 1$

Sub into ②: $4t + 2 = 3(2t+1) - 1$

$4t + 2 = 6t + 2 \rightarrow t = 0$, then $s = 2t+1 = 2(0)+1 = 1$

now sub those t and s into ③

if both sides are equal, then the path intersect

if not, the lines are skew

③: $3t + 5 = -5s + 10$

$3(0) + 5 = -5(1) + 10$

$5 = 5$ they are equal, so there is an intersection

at where? $\vec{r}_1(\overset{t}{0})$ or $\vec{r}_2(\overset{s}{1}) = \langle 3, 2, 5 \rangle$

intersection at $(3, 2, 5)$

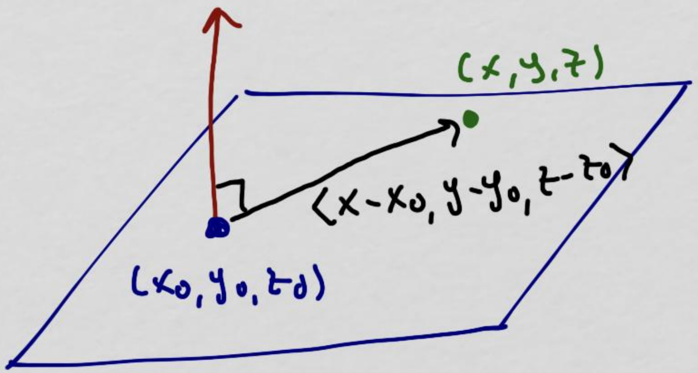
collision? No, because $t \neq s$

both objects go through $(3, 2, 5)$ but at different times.

now let's look at planes

a plane is defined by a vector orthogonal to it

$\vec{n}$: vector normal (orthogonal) to the plane



$\vec{n}$: vector normal to plane (normal vector)

(x_0, y_0, z_0) : some point the plane goes through

(x, y, z) : another point on the plane

note that as long as (x,y,z) is part of the plane,

$\langle x-x_0,\ y-y_0,\ z-z_0\rangle$ is always perpendicular to $\vec{n}$

let $\vec{n} = \langle a, b, c\rangle$

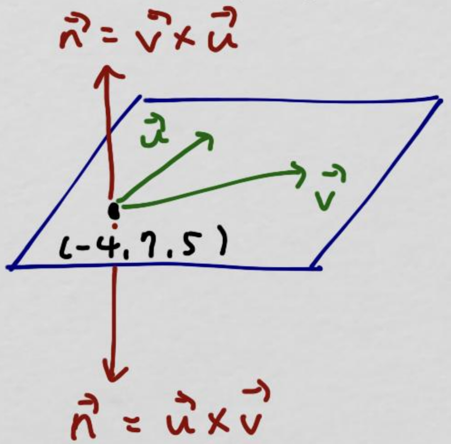
then $\boxed{\begin{array}{l} \langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0 \\ \text{or } \ a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \end{array}}$

equation of plane through (x_0, y_0, z_0) w/ normal vector $\langle a,b,c\rangle$

<u>example</u> Equation of plane containing the vectors

$\vec{u} = \langle 0, 1, 2 \rangle$ and $\vec{v} = \langle -1, -3, 0 \rangle$ and passes through

$(\overset{x_0}{-4}, \overset{y_0}{7}, \overset{z_0}{5})$



need normal vector (which is orthogonal to BOTH $\vec{u}$ and $\vec{v}$)

so, $\vec{n} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

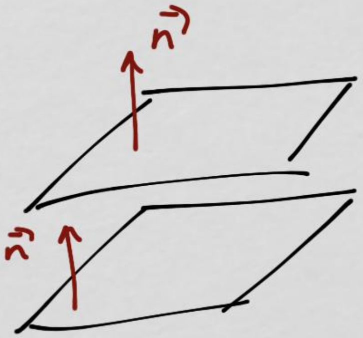
here, let's let $\vec{n} = \vec{v} \times \vec{u} = \langle -1, -3, 0 \rangle \times \langle 0, 1, 2 \rangle = \langle \underset{a}{-6}, \underset{b}{2}, \underset{c}{-1} \rangle$

then use $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$$\boxed{-6(x+4) + 2(y-7) - 1(z-5) = 0}$$

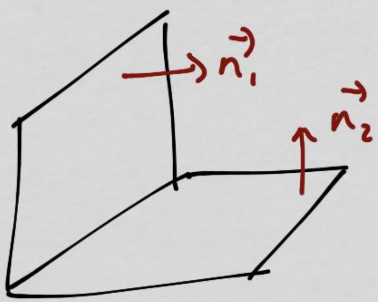
or $\boxed{-6x + 2y - z = 33}$

two planes are parallel if their normal vectors are parallel



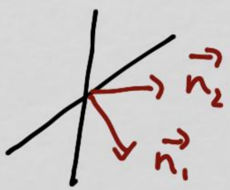
(floor and ceiling of your room)

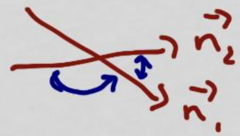
two planes are orthogonal if their normal vectors are orthogonal



(floor and a wall of your room)

the angle between two planes is the <u>smallest</u> angle between normal vectors





we use the smaller angle

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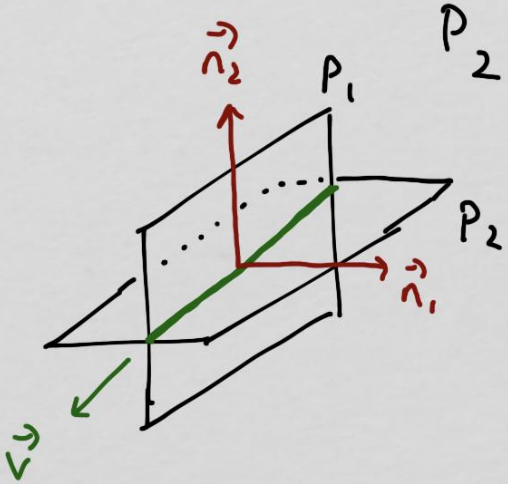
the intersection of two planes is a line (look at the place where the floor of your room meets a wall)

<u>example</u> Find the equation of the line of the intersection of the planes $P_1 : -2x - y + 4z = 1$ $\quad \vec{n}_1 = \langle -2, -1, 4 \rangle$

$P_2 : x + y + z = 1$ $\quad \vec{n}_2 = \langle 1, 1, 1 \rangle$



note the direction vector of the line is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$

so, $\vec{v} = \vec{n}_1 \times \vec{n}_2$ or $\vec{n}_2 \times \vec{n}_1$

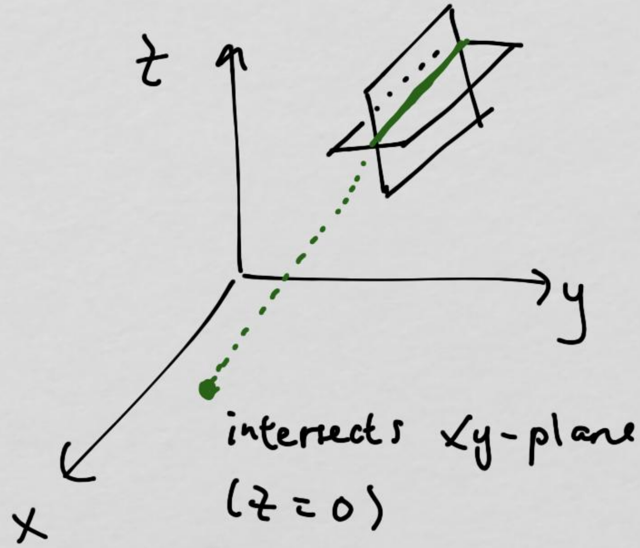
in this example, let's use $\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle -5, 6, -1 \rangle$

then we need a point on the line

this is a bit tricky

easiest way : find intersection of line with a coordinate plane

(xy-plane, xz-plane, or yz-plane)



let's find its intersection w/ xy-plane

$\rightarrow$ $z=0$ sub into plane equations

$P_1 : -2x-y+4z=1 \longrightarrow -2x-y=1$

$P_2 : x+y+z=1 \longrightarrow x+y=1$

solve $\begin{cases} -2x-y=1 \\ x+y=1 \end{cases}$ for x and y $\longrightarrow$ $x=-2,\ y=3$

so, the intersection of line w/ xy-plane is $(-2, 3, 0)$

therefore, its equation is $\boxed{\vec{r}(t) = \langle -2, 3, 0 \rangle + t \langle -5, 6, -1 \rangle}$

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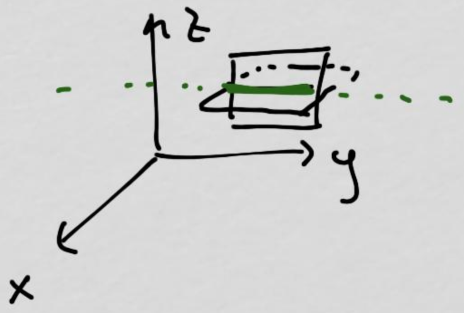
note that the line may not have an intersection with a chosen coordinate plane



never intersects xy-plane

if this were the case, there would be no solution in the system of equations from the planes

choose a different coordinate plane