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17.5 Curl and Divergence

we saw curl and divergence in 17.4 for 2D vector field.

In 17.5 we will look more general cases.

let's briefly revisit the gradient : $\vec{\nabla} \underset{\text{scalar}}{f} = \underbrace{\langle f_x , f_y \; f_z \rangle}_{\text{vector}}$

so, this means the del operator $\vec{\nabla}$ must behave like a vector

in fact, the del operator is defined as

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\left(\text{in 2D, } \vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)$$

it is a <u>vector-like</u> object but not a true vector because it doesn't have a magnitude and it doesn't have a direction

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by themselves, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ don't mean anything, they only take on a meaning when applied to a function

likewise, $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$ by itself is meaningless.

gradient: $\vec{\nabla} f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f$ (treat this like multiplying a vector by a scalar)

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle f_x, f_y, f_z \rangle$$

(that's what's really going on when we take a gradient)

Since $\vec{\nabla}$ is like a vector, we can cross and dot it with another vector.

$\vec{\nabla} \times \vec{F} = \text{curl } \vec{F}$

$\vec{\nabla} \cdot \vec{F} = \text{div } \vec{F}$

curl :
$$\boxed{\operatorname{curl}\vec{F} = \vec{\nabla} \times \vec{F}}$$

$$= \underbrace{\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle}_{\vec{\nabla}} \times \underbrace{\langle f, g, h \rangle}_{\vec{F}}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) - \vec{j}\left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) + \vec{k}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$= \langle h_y - g_z, f_z - h_x, g_x - f_y \rangle$$

if $\vec{F}$ is 2D, $\vec{F} = \langle f, g, 0 \rangle$, then from above formula

$$\operatorname{curl}\vec{F} = \vec{\nabla} \times \langle f, g, 0 \rangle = \langle 0, 0, g_x - f_y \rangle = (g_x - f_y)\vec{k}$$

(this was how we defined $\operatorname{curl}\vec{F}$ for a 2D $\vec{F}$ in 17.4)

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divergence : $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle$$

$$= \frac{\partial}{\partial x} f + \frac{\partial}{\partial y} g + \frac{\partial}{\partial z} h = f_x + g_y + h_z$$

in $\vec{F}$ is 2D , $\vec{F} = \langle f, g, 0 \rangle$

then from above, $\operatorname{div} \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, 0 \rangle$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

(this was how we defined divergence in 17.4)

now we have a general method to find curl and divergence

$\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}$

$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$

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example $\vec{F} = \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$

$$\text{curl } \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(x^2y^3z) - \frac{\partial}{\partial z}(x^3yz^2),\ -\left(\frac{\partial}{\partial x}(x^2y^3z) - \frac{\partial}{\partial z}(xy^2z^3)\right),\ \frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(xy^2z^3) \right\rangle$$

$$= \boxed{\langle 3x^2y^2z - 2x^3yz,\ 3xy^2z^2 - 2xy^3z,\ 3x^2yz^2 - 2xyz^3 \rangle}$$

$$\text{div } \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$$

$$= \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^3z)$$

$$= \boxed{y^2z^3 + x^3z^2 + x^2y^3}$$

don't forget: curl is vector
divergence is scalar

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we know that if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi = \langle \phi_x, \phi_y, \phi_z \rangle$

let's find out the curl of that

$$\text{curl } \vec{F} = \text{curl}(\vec{\nabla}\phi) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle \phi_x, \phi_y, \phi_z \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\phi_z - \frac{\partial}{\partial z}\phi_y, -\left(\frac{\partial}{\partial x}\phi_z - \frac{\partial}{\partial z}\phi_x\right), \frac{\partial}{\partial x}\phi_y - \frac{\partial}{\partial y}\phi_x \right\rangle$$

$$= \langle \underbrace{\phi_{zy} - \phi_{yz}}_{0}, \underbrace{\phi_{xz} - \phi_{zx}}_{0}, \underbrace{\phi_{yx} - \phi_{xy}}_{0} \rangle = \langle 0, 0, 0 \rangle = \vec{0}$$

so, the curl of the gradient of something is the zero vector

$\boxed{\text{the curl of a conservative vector field is the zero vector}}$

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if curl $\vec{F} = \vec{0}$, then $|\text{curl } \vec{F}| = 0 \rightarrow \vec{F}$ is irrotational

so, $\boxed{\text{a conservative vector field is irrotational}}$

does this agree with how we checked if $\vec{F} = \langle f, g, h \rangle$ is conservative?

old way: $\vec{F} = \langle f, g, h \rangle$ is conservative if $g_x = f_y$

$f_z = h_x$

$g_z = h_y$

now we know curl $\vec{F} = \langle \underline{\underline{h_y - g_z}}, \underline{\underline{f_z - h_x}}, \underline{\underline{g_x - f_y}} \rangle$

if 0, then conservative

which means curl $\vec{F} = \vec{0}$

so yes, it does agree with what we used in earlier sections.

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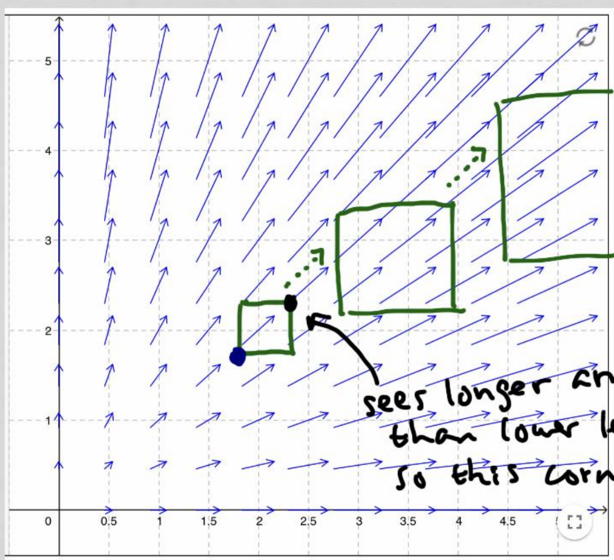
now we have a general way to check if $\vec{F}$ of any dimension is conservative : $\boxed{\text{if curl } \vec{F} = \vec{0}, \text{ then } \vec{F} \text{ is conservative}}$

now let's take a closer look at the divergence

divergence : the change in volume of a small element flowing in the vector field

if $\vec{F} = \langle f, g \rangle$ then $\text{div}\, \vec{F} = f_x + g_y$



$\rightarrow \vec{F} = \langle x, y \rangle \quad \text{div}\, \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2$

greater than 0
so a small box
grows as it
flows with the
field

sees longer arrow
than lower left
so this corner gets pulled
away from lower left

$\rightarrow$ growing box $\Leftrightarrow$ positive divergence

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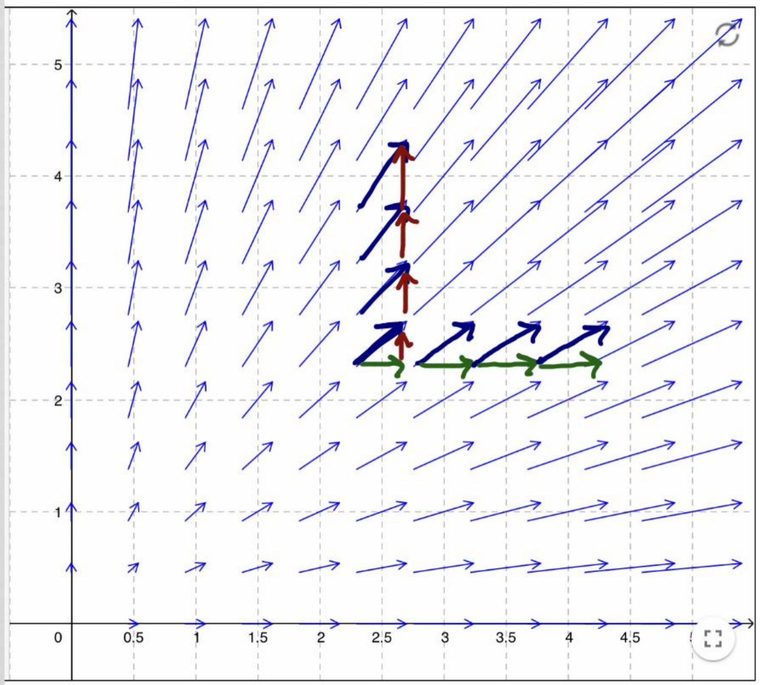

another way to come to the same conclusion:

$\operatorname{div}\vec{F} = f_x + g_y$

$\vec{F} = \langle f, g \rangle$

f_x : how the x-component changes as x increases

g_y : how the y-component changes as y increases

notice the green arrow (x-component of $\vec{F}$) is getting longer as we move right : $f_x > 0$

same thing happens with red arrow (y-component of $\vec{F}$) as we move up : $g_y > 0$

so, $f_x > 0$ and $g_y > 0$ give us $f_x + g_y > 0 \to \operatorname{div}\vec{F} > 0$

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we can now actually express Green's Theorem in the <u>vector form</u>

if $\vec{F} = \langle f, g \rangle$

then $$\oint_C f\,dx + g\,dy = \iint_R \underbrace{(g_x - f_y)}_{\text{z-component of curl } \vec{F}}\, dA$$

z-component of curl $\vec{F}$ can be written as $\text{curl}\,\vec{F} \cdot \vec{k}$

so, Green's Theorem can also be written as

$$\oint_C f\,dx + g\,dy = \iint_R \text{curl}\,\vec{F} \cdot \vec{k}\, dA$$

$\vec{k}$: perpendicular (normal) to a plane containing R

this is actually also a special case of Stokes' Theorem which we will see later.