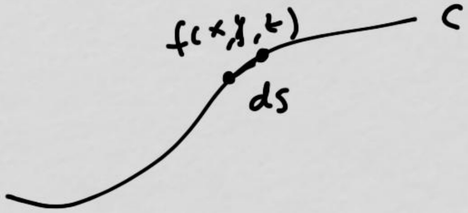
17.6 Surface Integrals (part 1)

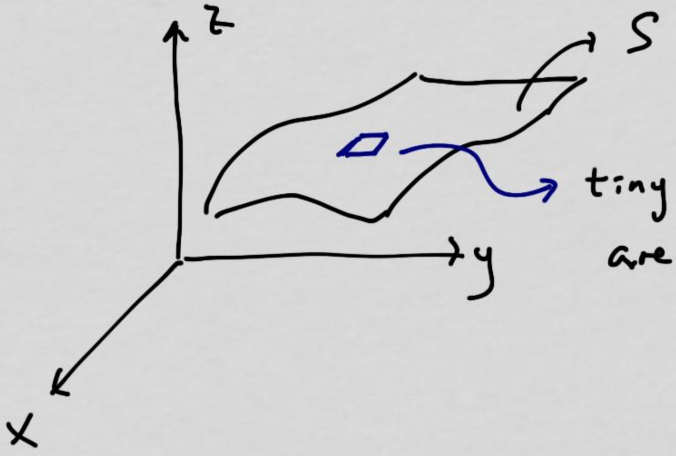
surface integral : extension of line integral

line integral of function : $\int_C f(x,y,z)\,ds$



accumulates $f(x,y,z)\,ds$ along the length of C

Surface integral is very similar : $\iint_S f(x,y,z)\,dS$ ← capital S

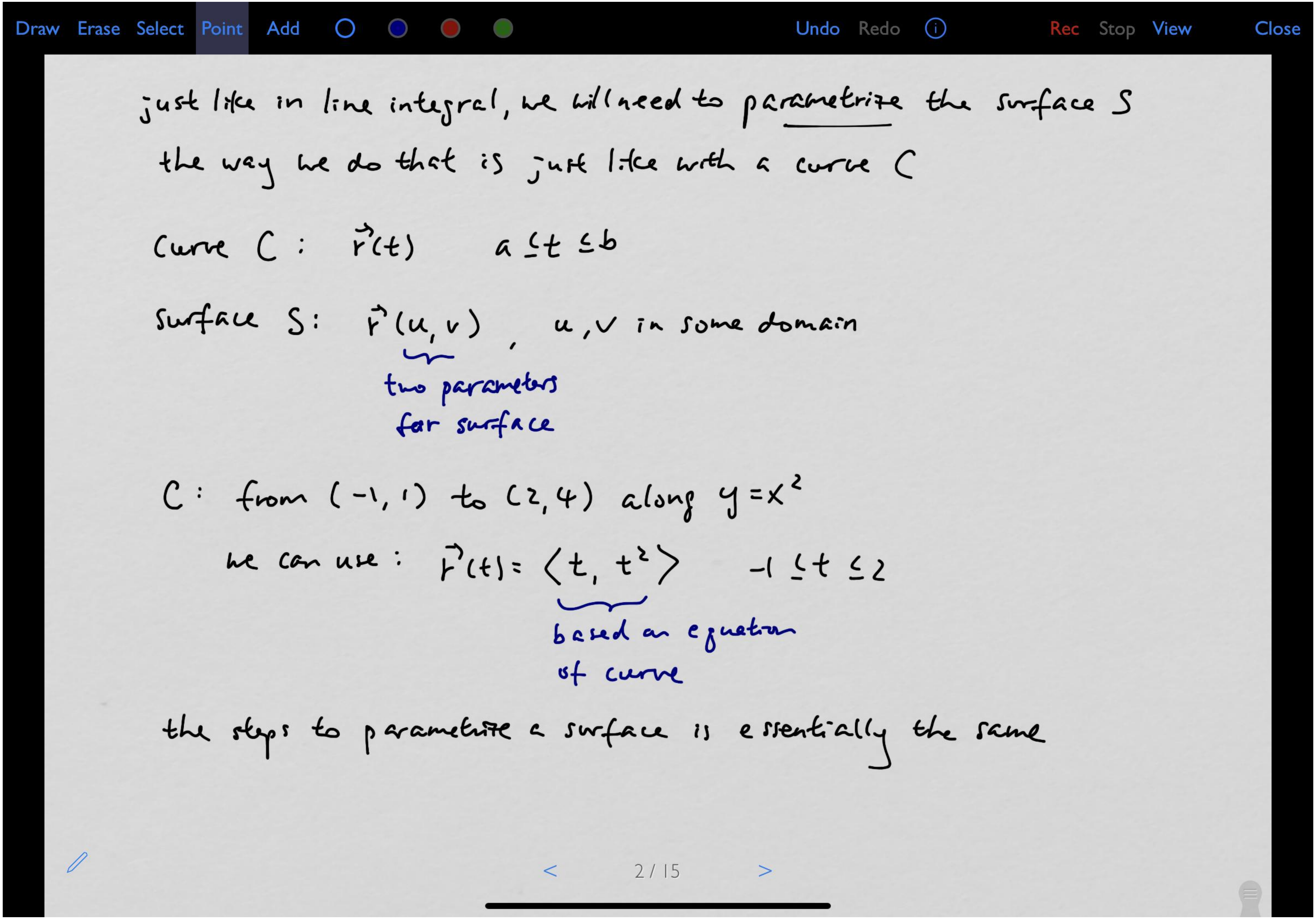


tiny piece of S

area of dS

the integral accumulates $f(x,y,z)$ all over the surface

(if $f(x,y,z)$ is density, then $\iint_S f(x,y,z)\,dS$ gives the mass of the surface)



just like in line integral, we will need to parametrize the surface S

the way we do that is just like with a curve C

Curve C : $\vec{r}(t)$ $\quad a \le t \le b$

Surface S: $\vec{r}(\underbrace{u,v}_{\text{two parameters for surface}})$, u, v in some domain

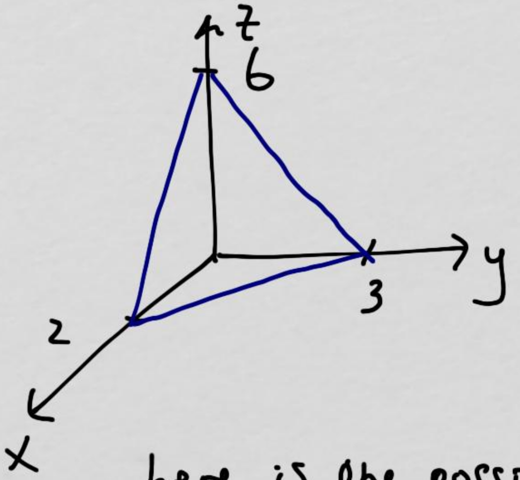
C : from (-1, 1) to (2, 4) along $y = x^2$

we can use : $\vec{r}(t) = \langle \underbrace{t, t^2}_{\text{based on equation of curve}} \rangle \quad -1 \le t \le 2$

the steps to parametrize a surface is essentially the same

example Parametrize the part of the plane $3x+2y+z=6$ in the first octant.

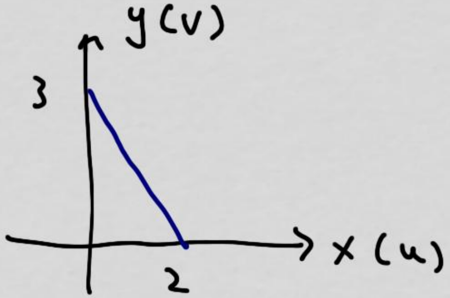
(parametrize and parameterize are both acceptable spellings)



goal: write $\vec{r}(u,v)$ that gives the position of each point of the surface

here is one possible parametrization

let $u=x$, $v=y$, then $z=6-3x-2y = 6-3u-2v$



$$0 \le u \le 2$$

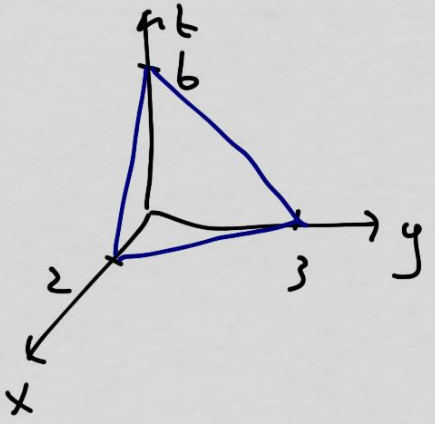
$$0 \le v \le 3 - \frac{3}{2}u$$

then each point of the surface has a position vector

$$\boxed{\vec{r}(u,v) = \{ \underset{x}{u}, \underset{y}{v}, \underbrace{6-3u-2v}_{z} \} \qquad 0 \le u \le 2, \quad 0 \le v \le 3 - \frac{3}{2}u}$$

(here, u, v are x and y, but in general, they don't have to be)

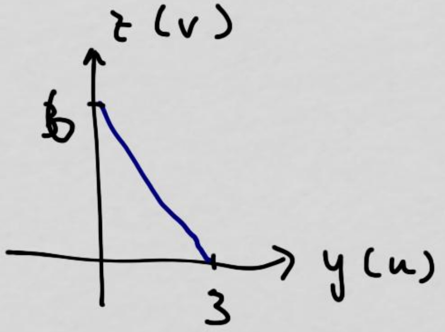
another possible parametrization



$$3x + 2y + z = 6$$

let $u = y$, $v = z$, then $x = 2 - \frac{2}{3}y - \frac{1}{3}z$

$$x = 2 - \frac{2}{3}u - \frac{1}{3}v$$



$$0 \le u \le 3$$

$$0 \le v \le 6 - 2u$$

each point's position

$$\boxed{\vec{r}(u,v) = \left\{ \underbrace{2-\frac{2}{3}u-\frac{1}{3}v}_{x},\ \underset{y}{\underset{\uparrow}{u}},\ \underset{z}{\underset{\uparrow}{v}} \right\} \quad 0 \le u \le 3, \quad 0 \le v \le 6-2u}$$

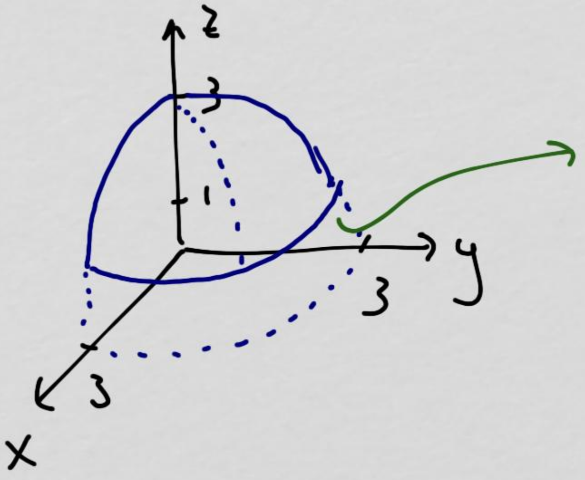
of course, the parametrization is not unique

u, v can represent the variables of a coordinate system

(or functions of them)

(they don't have to have physical meanings, but often they do)

example $x^2+y^2+z^2=9$ in the first octant, $1 \le z \le 3$

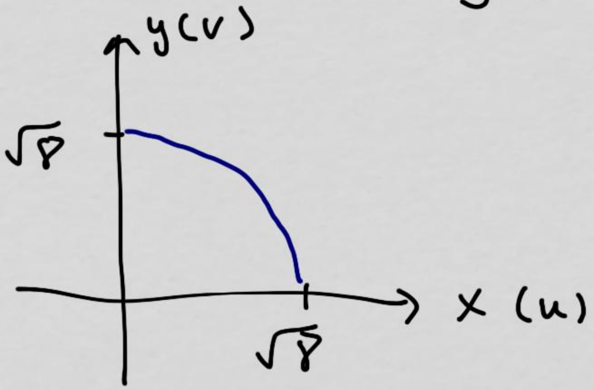


this circle, when projected down onto xy-plane, has radius $\sqrt{8}$

$z=1, \quad x^2+y^2+1=9 \to x^2+y^2=8$

one possibility is to let u, v be two of the variables

let $u=x$, $v=y$, then $\underline{z=\sqrt{9-x^2-y^2}=\sqrt{9-u^2-v^2}}$ based on shape



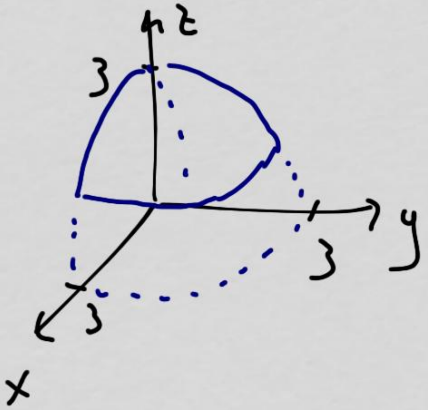
$0 \le u \le \sqrt{8}$

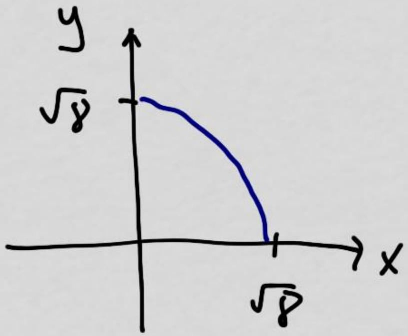
$0 \le v \le \sqrt{8-u^2}$

now the position vector of each point is

$$\boxed{\begin{array}{c} \vec{r}(u,v) = \Big\langle \underset{x}{u}\,,\ \underset{y}{v}\,,\ \underbrace{\sqrt{9-u^2-v^2}}_{z} \Big\rangle \\ \\ 0 \le u \le \sqrt{8} \quad , \quad 0 \le v \le \sqrt{8-u^2} \end{array}}$$

another possiblity: use cylindrical coordinates





$$0 \le r \le \sqrt{8}$$

$$0 \le \theta \le \pi/2$$

$$z = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$$

now we can let $u = r$, $v = \theta$

$$x = r\cos\theta = u\cos v$$

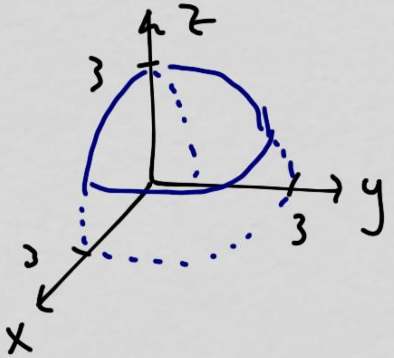
$$y = r\sin\theta = u\sin v$$

$$z = z = \sqrt{9 - r^2}$$

so,

$$\boxed{\begin{gathered}\vec{r}(u,v) = \left\langle u\cos v,\ u\sin v,\ \sqrt{9-u^2}\right\rangle \\ 0 \le u \le \sqrt{8}, \quad 0 \le v \le \pi/2\end{gathered}}$$

or use spherical coordinates

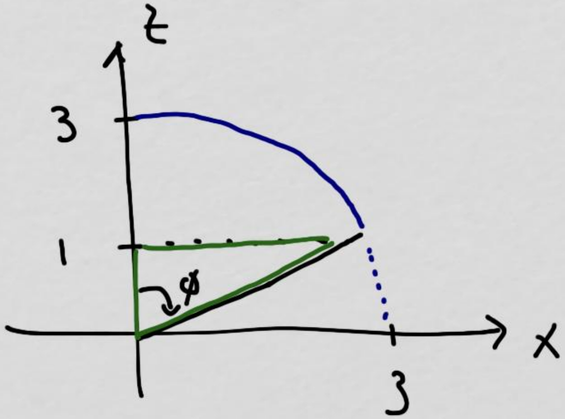


$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$

$\rho = 3$ because this is part of surface of sphere radius 3

let $u = \theta$, → $0 \le u \le \pi/2$

let $v = \phi$



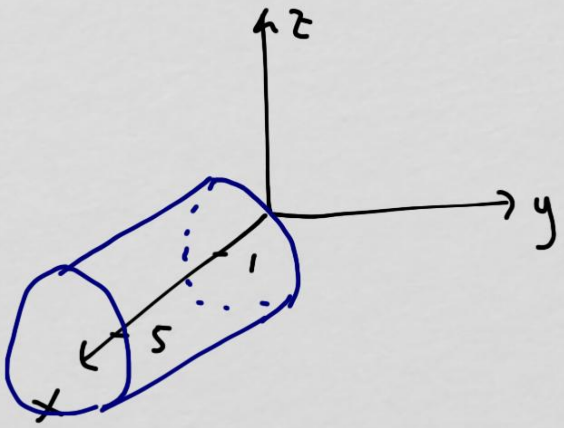
so, $\cos\phi = \frac{1}{3}$ $\quad \phi = \cos^{-1}\left(\frac{1}{3}\right)$

so, $\quad 0 \le v \le \cos^{-1}\left(\frac{1}{3}\right)$

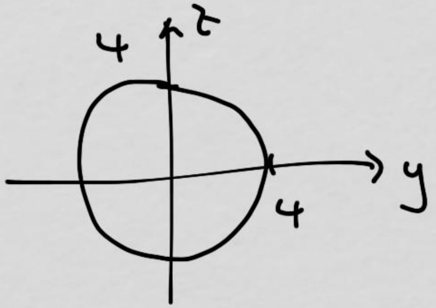
parametrization:

$$\boxed{\begin{array}{l} \vec{r}(\underset{\theta}{u}, \underset{\phi}{v}) = \langle \underbrace{3\sin v \cos u}_{\rho \sin\phi\cos\theta}, \underbrace{3\sin v \sin\theta}_{\rho\sin\phi\sin\theta}, \underbrace{3\cos v}_{\rho\cos\phi} \} \\ \\ \qquad 0 \le u \le \pi/2, \qquad 0 \le v \le \cos^{-1}\left(\frac{1}{3}\right) \end{array}}$$

example $y^2+z^2=16$ $\quad 1 \le x \le 5$



cylinder, so let's use cylindrical



so, $r=4$ (just surface of cylinder)

$0 \le \theta \le 2\pi$

$y = r\cos\theta = 4\cos\theta$

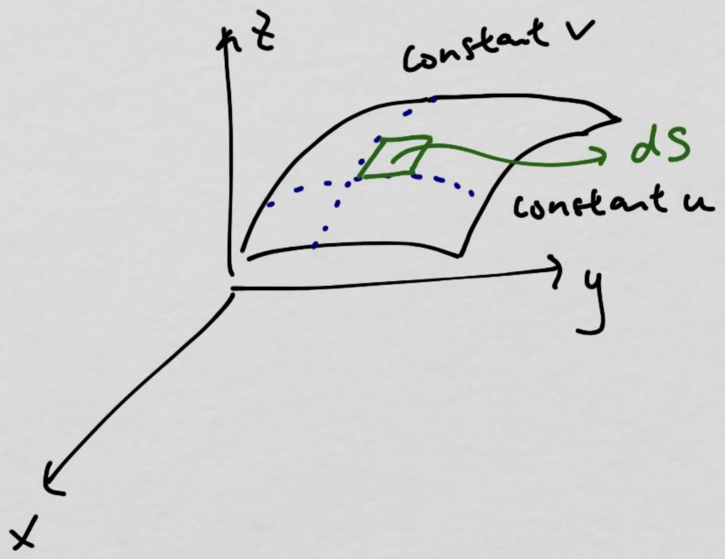
plays the role that x usually does (horiz. part of polar)

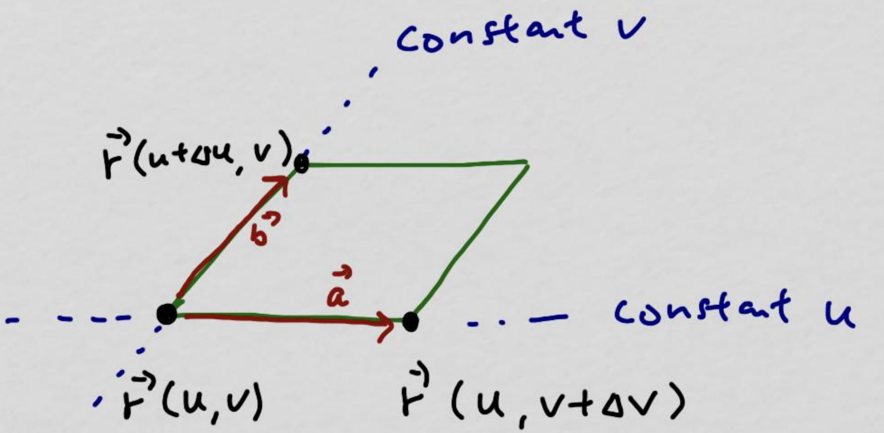
$z = r\sin\theta = 4\sin\theta$

let $u = \theta$ $\quad v = x$

$$\boxed{\vec{r}(u,v) = \langle \underset{x}{V}, 4\cos u, 4\sin u \rangle \quad 0 \le u \le 2\pi, \; 1 \le v \le 5}$$

now let's look at dS in $\iint_S f \, \underbrace{dS}_{\text{surface area of a tiny patch of } S}$





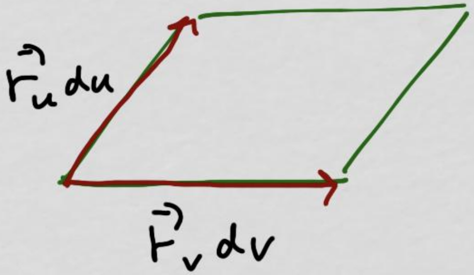
$$\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u,v)$$

$\boxed{\vec{a} = \vec{r}_v \, dv}$ $(\Delta v \to dv)$

similarly, $\boxed{\vec{b} = \vec{r}_u \, du}$

recall $\dfrac{\partial f}{\partial x} = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x,y)}{h}$

then $f(x+h, y) - f(x,y) = f_x h$
(when $h \to 0$)



area of little patch (and dS) is

$$|\vec{r}_u \times \vec{r}_v| \, du\,dv$$

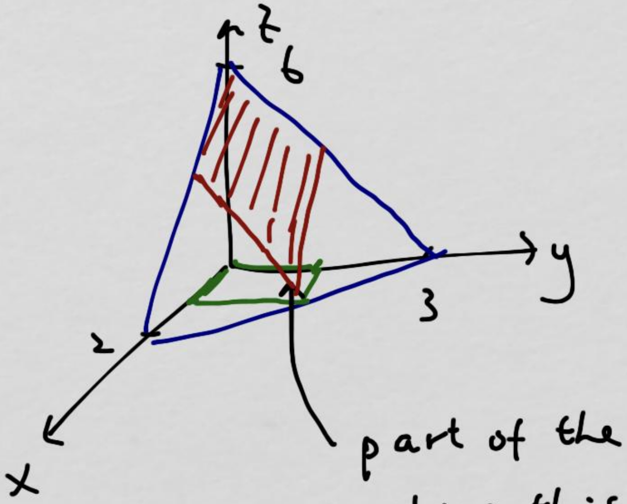
so, $\boxed{dS = |\vec{r}_u \times \vec{r}_v| \, \underbrace{du\,dv}}$

or the other around, depending on what is appropriate

now we can calculate $\iint_S f\,dS$ (or $\iint_S ds$ to find area)

<u>example</u> $\iint_S (x+y)\,dS$

S: part of the plane $3x+2y+z=6$ in the first octant above $0 \le x \le 1$, $0 \le y \le 3/2$



part of the plane above this

let's reuse the parametrization from earlier:

$$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$$

here, $0 \le u \le 1$, $0 \le v \le 3/2$

$$\vec{r}_u = \langle 1, 0, -3 \rangle$$

$$\vec{r}_v = \langle 0, 1, -2 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{14}$$

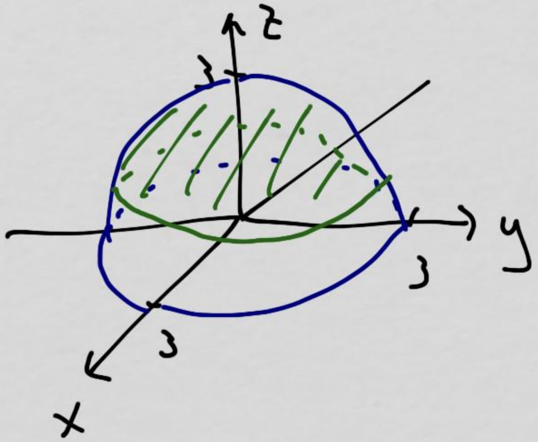
$$dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$$\boxed{dS = \sqrt{14}\,du\,dv}$$

$$\iint_S (x+y)\,dS = \underset{v}{\int_0^{3/2}} \underset{u}{\int_0^{1}} \underbrace{(u+v)}_{x+y} \underbrace{\sqrt{14}\,du}_{dS}\,dv = \cdots = \frac{15\sqrt{14}}{8}$$

example Surface area of a spherical cap cut from a sphere of radius 3

$$3/2 \le z \le 3$$



let's again reuse parametrization from earlier

$$\vec{r}(\underset{\theta}{u},\underset{\phi}{v}) = \langle 3\sin v\cos u,\ 3\sin v\sin u,\ 3\cos v\rangle$$

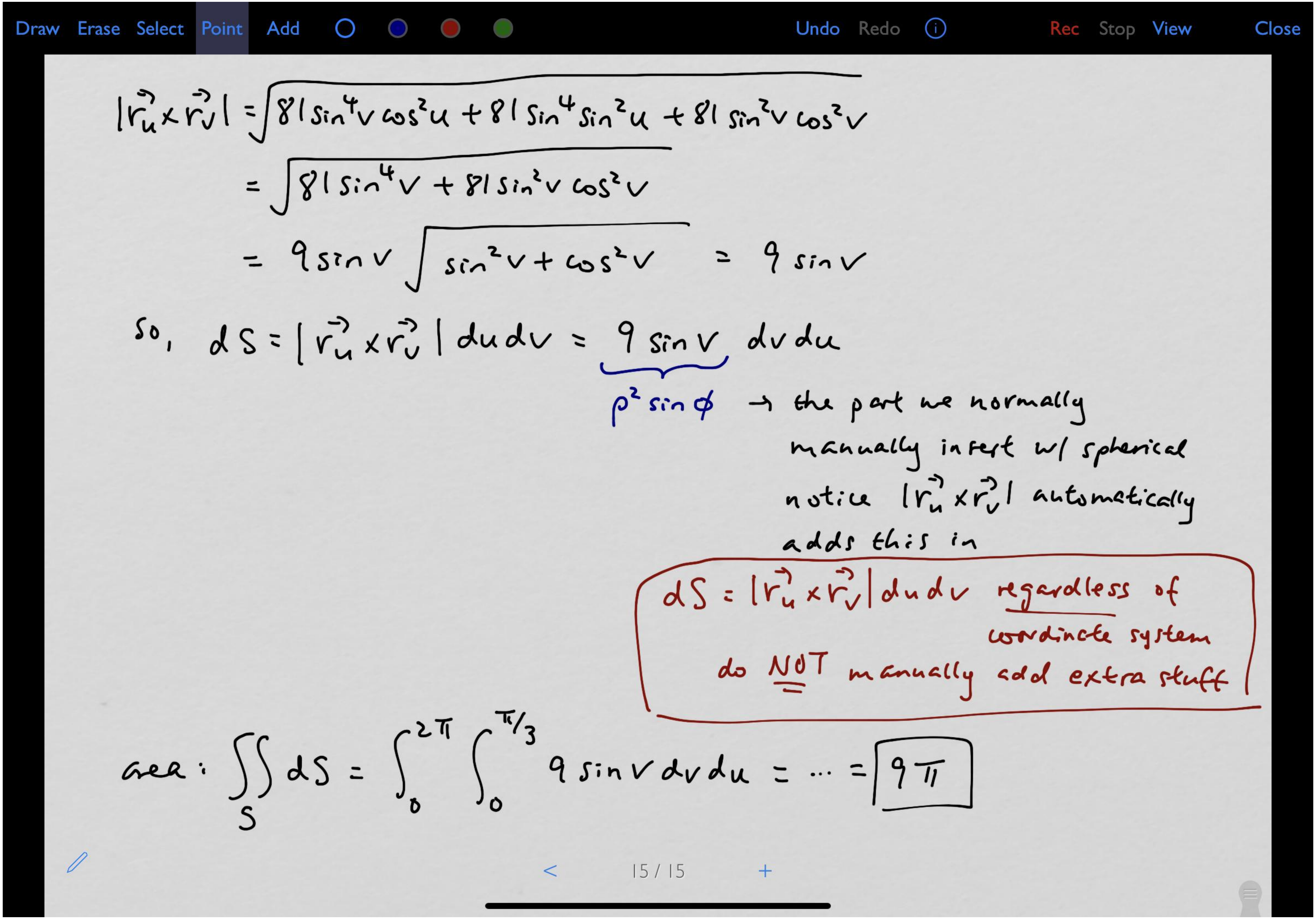
$$0 \le u \le 2\pi, \qquad 0 \le \phi \le \pi/3$$

end w/ bottom at $z = 3/2$

$$\vec{r}_u = \langle -3\sin v\sin u,\ 3\sin v\cos u,\ 0\rangle$$

$$\vec{r}_v = \langle 3\cos v\cos u,\ 3\cos v\sin u,\ -3\sin v\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -9\sin^2 v\cos u,\ -9\sin^2 v\sin u,\ 9\sin v\cos v\rangle$$



$$|\vec{r}_u \times \vec{r}_v| = \sqrt{81\sin^4 v\cos^2 u + 81\sin^4 \sin^2 u + 81\sin^2 v\cos^2 v}$$

$$= \sqrt{81\sin^4 v + 81\sin^2 v\cos^2 v}$$

$$= 9\sin v\sqrt{\sin^2 v + \cos^2 v} = 9\sin v$$

so, $dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv = \underbrace{9\sin v}_{\rho^2\sin\phi}\,dv\,du$

→ the part we normally manually insert w/ spherical

notice $|\vec{r}_u \times \vec{r}_v|$ automatically adds this in

$dS = |\vec{r}_u \times \vec{r}_v|\,du\,dv$ regardless of coordinate system

do NOT manually add extra stuff

area: $\iint_S dS = \int_0^{2\pi}\int_0^{\pi/3} 9\sin v\,dv\,du = \cdots = \boxed{9\pi}$