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17.6 Surface Integrals (part 2)

last time: $\iint_S f(x,y,z)\,dS = \iint_S f(x,y,z)\,|\vec{r}_u \times \vec{r}_v|\,dA$

where the surface is parametrized as $\vec{r}(u,v)$ u,v in R.

Sometimes, it might be more convenient to calculate the integral without parametrizing the surface, especially if the surface is already described explicitly $\to$ $z = f(x,y)$

if that is the case, $\vec{r}(x,y) = \langle x, y, z \rangle$ $\leftarrow$ $z = f(x,y)$

$u \nearrow$ $\nwarrow v$

$\vec{r}_u \nearrow \vec{r}_x = \langle 1, 0, z_x \rangle$

$\vec{r}_v \nearrow \vec{r}_y = \langle 0, 1, z_y \rangle$

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$$\vec{r}_x \times \vec{r}_y = \langle -z_x, -z_y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + z_x^2 + z_y^2}$$

therefore, $\iint_S f(x,y,z)\,dS$ becomes $\iint_R f(x,y,z) \underbrace{\sqrt{1+z_x^2+z_y^2}\,dA}_{dS}$

$dA = dx\,dy = dy\,dx$

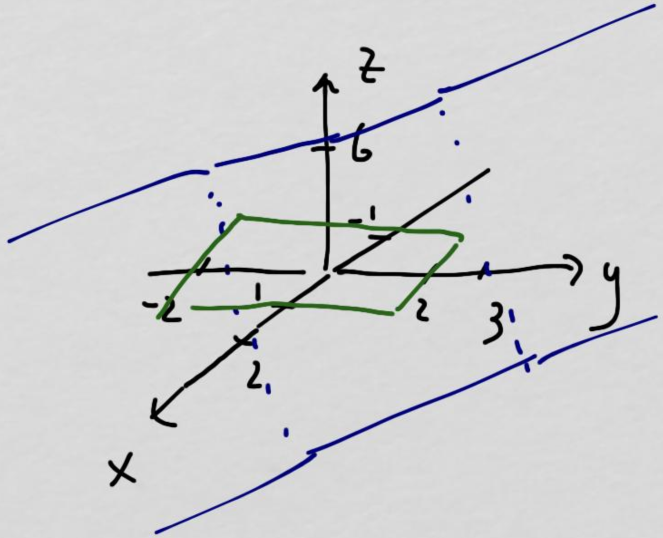
this formula might be convenient if $z = f(x,y)$ is already given

and Cartesian coordinates are used.

example

$$\iint_S (x+y)\,dS$$

S: the plane $z = 6-3x-2y$ above the region $|x| \le 1,\ |y| \le 2$



integrate $(x+y)$ on the plane above the green rectangle.

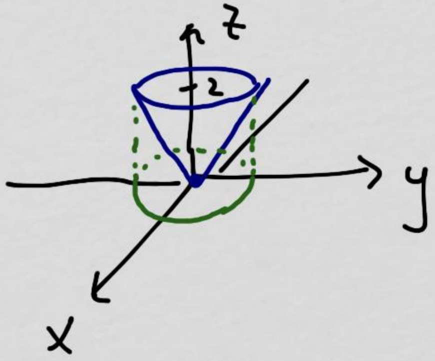
$$dS = \sqrt{1+z_x^2+z_y^2}\,dA$$

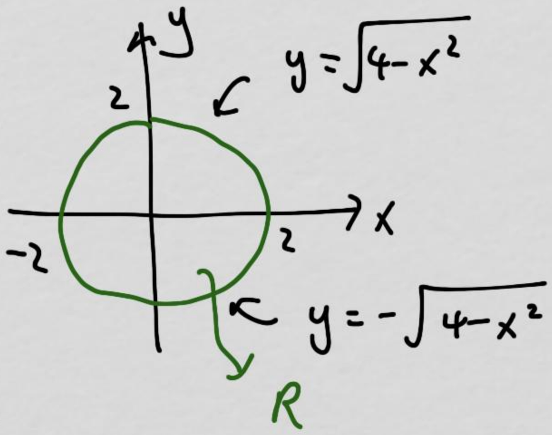
$$= \sqrt{1+(-3)^2+(-2)^2}\,dA$$

$$= \sqrt{14}\,dA = \sqrt{14}\,dy\,dx = \sqrt{14}\,dx\,dy$$

$$\int_{-1}^{1}\int_{-2}^{2} (x+y)\sqrt{14}\,dy\,dx = \cdots = \boxed{0}$$

<u>example</u> Find surface area of $z^2 = x^2 + y^2$, $0 \le z \le 2$





$$z = \sqrt{x^2 + y^2}$$

$$z_x = \frac{x}{\sqrt{x^2+y^2}}$$

$$z_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$dS = \sqrt{1 + z_x^2 + z_y^2}\, dA$$

$$= \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dA$$

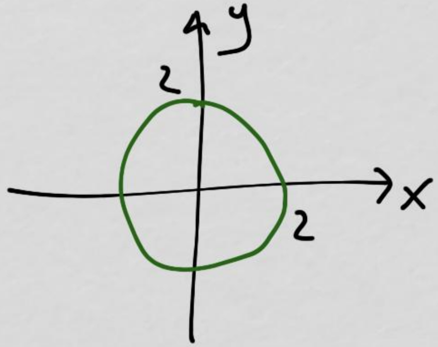
R: $-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$$\iint_S dS = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\, dy\, dx$$

this can be "calculate" this double integral by using geometry

$$\sqrt{2}\underbrace{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx}_{\text{area of } R}$$



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

area of R

circle of radius 2

area $= \pi(2)^2 = 4\pi$

$$= \boxed{\sqrt{2}\cdot 4\pi}$$

lateral surface area of cone

but, of course, we could also have changed to polar

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\,dy\,dx = \int_{0}^{2}\int_{0}^{2\pi} \sqrt{2}\,r\,dr\,d\theta = \dots = \boxed{4\sqrt{2}\pi}$$

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note we had to insert the "r" in dA

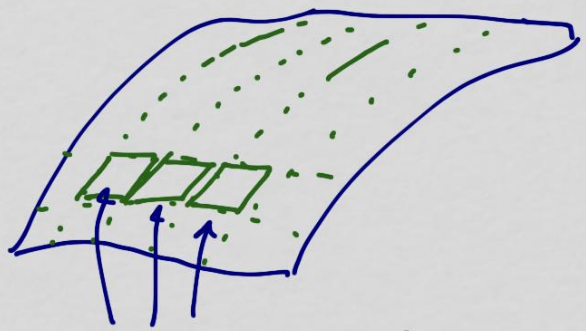
because we started with a Cartesian integral and then changed to polar.

(and the formula $\iint_R \sqrt{1+z_x^2+z_y^2}\,dA$ assumes Cartesian)

if we had parametrized the surface, then $\iint_S |\vec{r}_u \times \vec{r}_v|\,dA$

automatically provide the pieces needed for whatever the coordinate system is $\rightarrow$ do NOT manually add things (e.g. r in dA of polar) we can calculate the average value of $f(x,y,z)$ on S by doing something similar to how we calculated averages in the past

$\iint\limits_S f(x,y,z)\,dS$ is the total of the accumulation of $f(x,y,z)$ all over the surface



accumulate $f(x,y,z)\,dS$ all over.

by the definition of average, if $f(x,y,z) = \text{constant} = f_{avg}$ everywhere, then $\iint\limits_S f_{avg}\,dS$ would give the same result.

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this means $\iint_S f(x,y,z)\,dS = \iint_S f_{avg}\,dS$

constant, can come out of integral

$$= f_{avg} \underbrace{\iint_S dS}_{\text{surface area of } S}$$

so, $\boxed{f_{avg} = \dfrac{\iint_S f(x,y,z)\,dS}{\iint_S dS}}$

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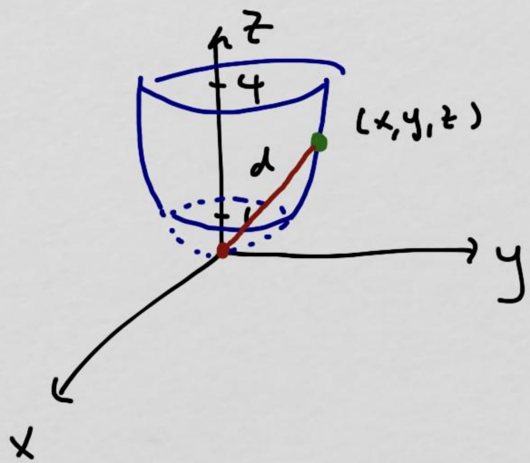
example Find the average distance from the origin of the surface $z = x^2 + y^2$, $1 \le z \le 4$

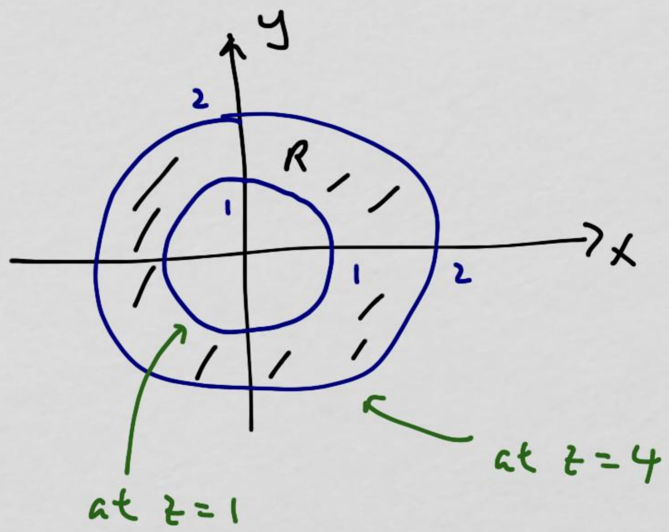


we want to find the average of the function $d = f(x,y,z)$

$$= \sqrt{x^2+y^2+z^2}$$

so, $$f_{avg} = \frac{\iint_S f(x,y,z)\, dS}{\iint_S dS}$$

$$dS = \sqrt{1 + z_x^2 + z_y^2}\, dA = \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4(x^2+y^2)}\, dA$$



Polar is obviously better

$$1 \leq r \leq 2$$

$$0 \leq \theta \leq 2\pi$$

$$dA = \underline{\underline{r}}\, dr\, d\theta$$

we need to supply this

$$\iint_S f(x,y,z)\, dS = \int_0^{2\pi}\int_1^2 \sqrt{r^2+(r^2)^2}\ \underbrace{\sqrt{1+4r^2}\ r\, dr\, d\theta}_{dS}$$

x^2+y^2 $\qquad$ $z = x^2+y^2$

$$= \cdots = 2\pi\,(15.208)$$

$$\iint_S dS = \int_0^{2\pi}\int_1^2 \sqrt{1+4r^2}\, r\,dr\,d\theta = \cdots = 2\pi\,(4.509)$$

therefore, the average distance is

$$f_{avg} = \frac{\iint_S f(x,y,z)\,dS}{\iint_S dS} = \frac{2\pi\,(15.208)}{2\pi\,(4.509)} \approx 3.1$$

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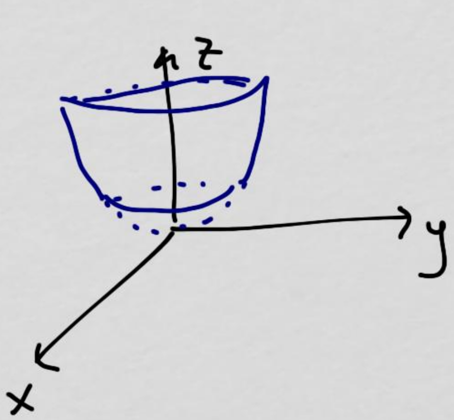
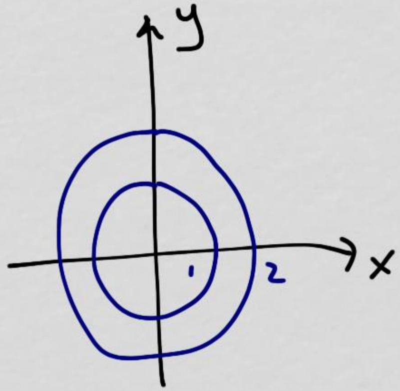
what if we had parametrized the surface?





we would see R and decide polar is good for the area

$1 \le r \le 2$

$0 \le \theta \le 2\pi$

$x = r\cos\theta, \quad y = r\sin\theta$

$z = x^2 + y^2 = r^2$

let $u = r, \quad v = \theta$

$\vec{r}(u,v) = \langle u\cos v, \ u\sin v, \ u^2 \rangle$

$1 \le u \le 2$
$0 \le v \le 2\pi$

$\vec{r}_u = \langle \cos v, \sin v, 2u \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -2u^2\cos v, \ -2u^2\sin v, \ u \rangle$

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$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4u^4\cos^2 v + 4u^4 \sin^2 v + u^2} = \sqrt{4u^4+u^2} = \underline{\underline{u\sqrt{1+4u^2}}}$$

$$\iint_S f\, dS = \int_0^{2\pi}\int_1^2 \sqrt{u^2+u^4}\cdot \underline{\underline{\sqrt{1+4u^2}\, u}}\, du\, dv$$

$$\sqrt{x^2+y^2+z^2}$$

$$= \sqrt{r^2+r^4}$$

$$= \sqrt{u^2+u^4}$$

notice $|\vec{r}_u \times \vec{r}_v|$ process automatically provides this

$$= \cdots = 2\pi\ (15.208)$$

and $\iint_S dS$ is done the same way.