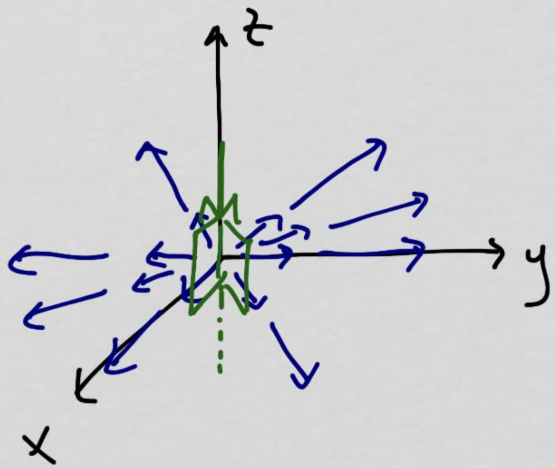
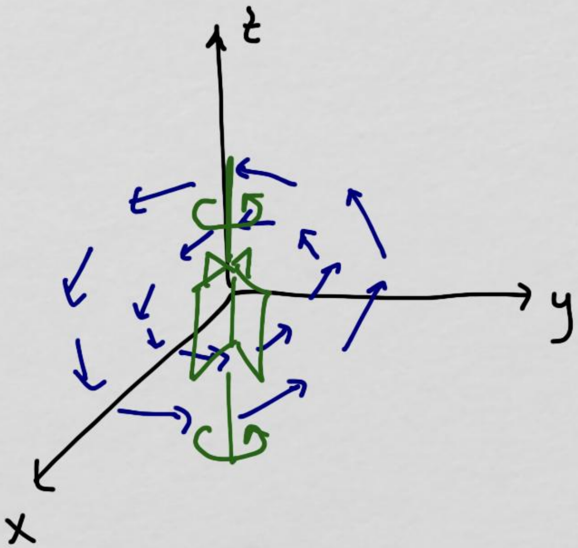
17.7 Stokes' Theorem (part 2)

more about curl of a vector field

let $\vec{F} = \langle x, y, 0 \rangle$

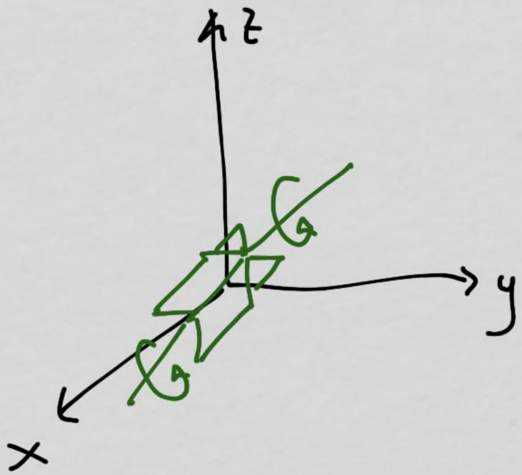


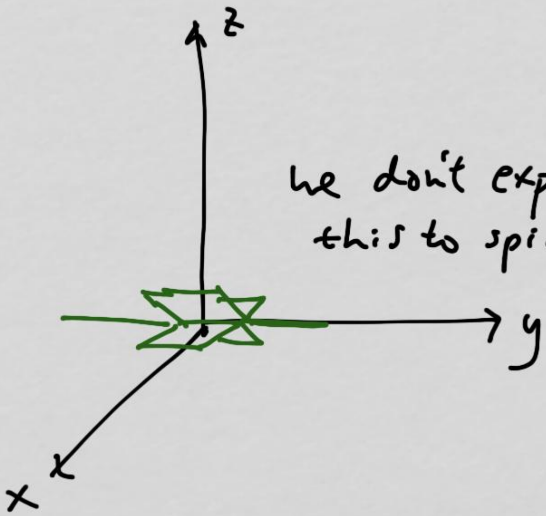
if a paddle wheel with axis along the z-axis is placed at the origin, we don't expect the wheel to turn because the field is pushing on all parts of the paddle wheel with the same flow now look at $\vec{F} = \langle -y, x, 0 \rangle$



now, it's easy to see that if a paddle wheel is placed at origin with axis along the z-axis, the wheel would spin.

and it would spin with less angular speed if the axis is tilted





we don't expect this to spin at all

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we can get all this from looking at the curl of the vector field

$\vec{F} = \langle x, y, 0 \rangle \qquad \text{curl } \vec{F} = \vec{\partial} \times \vec{F} = \langle 0, 0, 0 \rangle$

$\vec{F} = \langle -y, x, 0 \rangle \qquad \text{curl } \vec{F} = \langle 0, 0, 2 \rangle$

this tells us that if the axis
of the paddle wheel is along z-axis,
then it would spin with max speed
(because z-component has the
largest magnitude)
if placed w/ the axis along x- or y-axis
there would be no spin (0 in x- or y-
components)
the z-component is constant, so the spin rate
would be constant, no matter where the
paddle wheel is, as long as the axis stays put.

$$\vec{F} = \langle 5 - z^2, 0, 0 \rangle$$

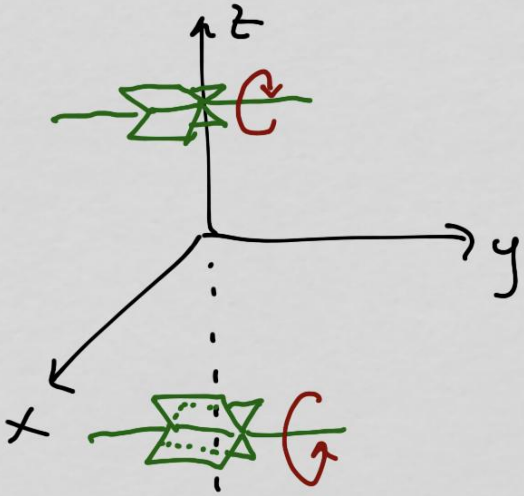
$$\text{curl } \vec{F} = \langle 0, -2z, 0 \rangle$$

maximum spin rate if axis is along the y-axis

and the spin rate increases if $|z|$ increases

(the higher $|z|$ is, the larger the magnitude of the y-component of curl)

furthermore, the negative sign tells us the spin would be against the right hand rule ($\vec{j} = \vec{k} \times \vec{i}$, so "normal" spin is counterclock when viewed with y-axis coming out of page. Here, the negative sign makes the spin go the other way)



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when we calculate $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$, we are essentially accumulating the spin of the paddle wheels everywhere on the surface by orienting the axis along $\vec{n}$

we know that if $\vec{F}$ is conservative, then $\text{curl}\,\vec{F} = \vec{0}$

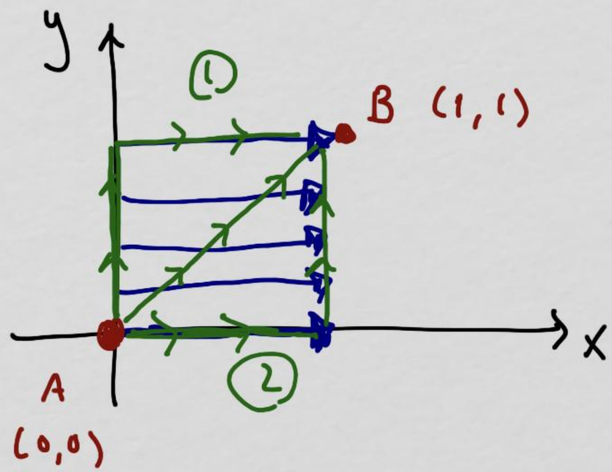
(a conservative vector field is irrotational)

but why?

from the Fundamental Theorem of Line Integrals, if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi$ and $\int_C \vec{F}\cdot d\vec{r} = \underset{\text{end}}{\phi(B)} - \underset{\text{start}}{\phi(A)}$

this means that the accumulated tangential $\vec{F}$ along C is the same regardless of the path

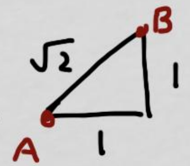
$\vec{F} : \langle 1, 0 \rangle$



one way to go from A to B is along ①
: up (no accumulated tangential component)
the right (accumulating $\vec{F} \cdot d\vec{r}$)

along ② : right (accumulate $\vec{F} \cdot d\vec{r}$) then
up (no accumulation)

the accumulation is clearly the
same along these two paths

along ③ : the distance is $\sqrt{2}$ () and the accumulated

amount of vector field $\vec{F} \cdot d\vec{r}$ along ③ is $\frac{1}{\sqrt{2}} |\vec{F}| = \frac{1}{\sqrt{2}}$

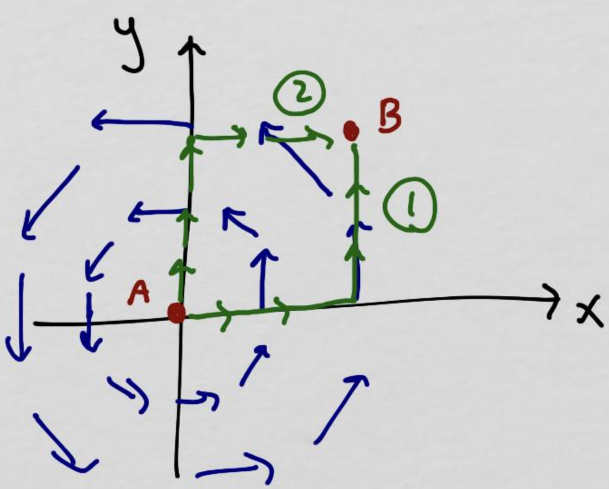
because ③ takes 45° angle to go from A to B

so, the total accumulation is $\frac{1}{\sqrt{2}} \cdot \sqrt{2} = 1$

(same as that of ① or ②)

if $\vec{F}$ is NOT conservative, for example

$\vec{F} = \langle y, x \rangle$



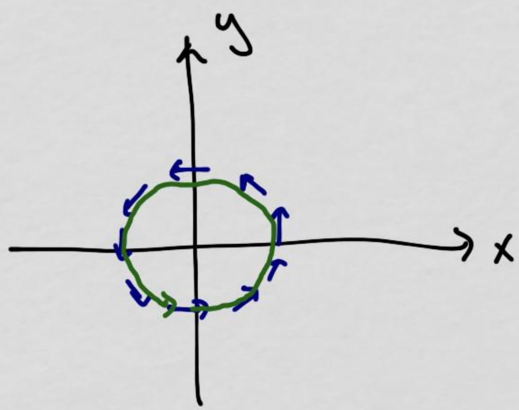
on ①, first half has no accumulation, because $\vec{F}$ is perpendicular to path then accumulate positive amount on 2nd half (it's with the rotation)

on ②, first half has no accumulation but the 2nd half has negative accumulation because it is against the rotation

① : "free ride" on 2nd half

② : fighting the flow (not free) on 2nd half

so, $\int_C \vec{F} \cdot d\vec{r}$ is path-dependent in a non-conservative field



going counterclockwise, free ride the entire time

going clockwise, must fight the flow

clearly very different $\int_C \vec{F}\cdot d\vec{r}$

this is a feature of non-conservative field

example $S: z^2 = a^2(1-x^2-y^2)$, a is a positive number, $z \geq 0$

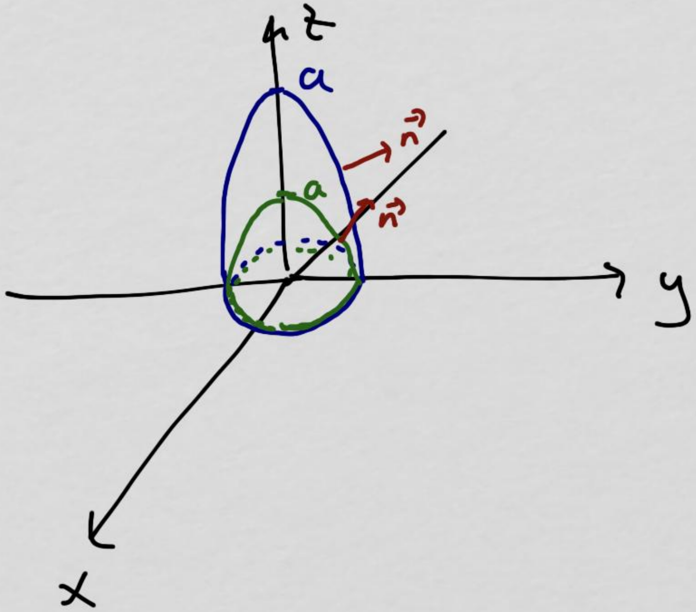
$\vec{n}$ to be upward

$\vec{F} = \langle x-y,\ y+z,\ z-x \rangle$

Find a such that $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$ is maximized.

shape? $z^2 = a^2 - a^2x^2 - a^2y^2$

$a^2x^2 + a^2y^2 + z^2 = a^2 \rightarrow x^2 + y^2 + \dfrac{z^2}{a^2} = 1$ ellipsoid



for $z = k$, we have circle cross-sections, but the z-axis length is affected by a

changing a clearly changes $\vec{n}$, so it is not unreasonable to expect $\operatorname{curl}\vec{F}\cdot\vec{n}$ to change, so changing $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$

however, the shape is symmetric, so there might be cancellations

But Stokes' Theorem tells us that only the boundary curve matters, and <u>all</u> surfaces with the same boundary curve have the same $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$

Therefore, the value of $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$ should be independent of a

Verify.

we have several choices:

1) parametrize the ellipsoid, then calculate $\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$

2) calculate $\oint_C \vec{F} \cdot d\vec{r}$ on the boundary (at $z=0$)

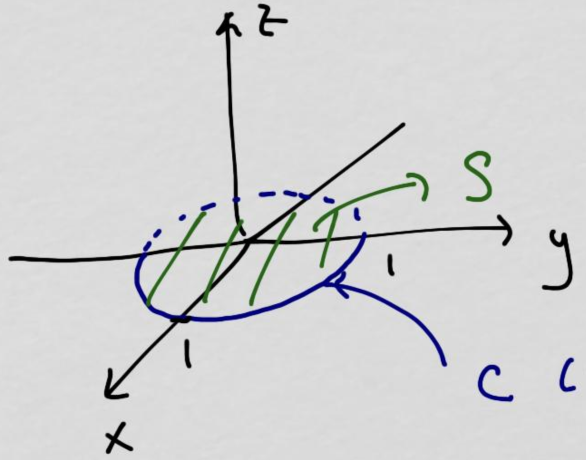
3) use a simpler surface w/ the same boundary and calculate $\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$

whatever we do, check curl $\vec{F}$ first (if curl $\vec{F} = \vec{0}$, then we are done)

$\vec{F} = \langle x-y, y+z, z-x \rangle$

$\operatorname{curl} \vec{F} = \langle 1, 1, 1 \rangle$

not $\vec{0}$, but very simple $\rightarrow$ choice 3) might be the best



$$z^2 = a^2(1 - x^2 - y^2)$$

at $z=0$, $1 - x^2 - y^2 = 0 \rightarrow x^2 + y^2 = 1$

S : disk of radius 1 at $z=0$

$$\vec{r}(\underset{r}{u}, \underset{\theta}{v}) = \langle u\cos v,\ u\sin v,\ 0\rangle \qquad 0 \le u \le 1 \qquad 0 \le v \le 2\pi$$

$$\left.\begin{aligned} \vec{r}_u &= \langle \cos v, \sin v, 0\rangle \\ \vec{r}_v &= \langle -u\sin v, u\cos v, 0\rangle \end{aligned}\right\} \quad \vec{r}_u \times \vec{r}_v = \langle 0, 0, u\rangle$$

is this upward?
yes, because
$u \ge 0$

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^1 \langle 1,1,1\rangle \cdot \langle 0,0,u\rangle \, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 u\, du\, dv = \int_0^{2\pi} \frac{1}{2}\, dv = \boxed{\pi}$$

So, that confirms that the flux of the curl is independent of a.

Be careful with apply Stokes' Theorem. It is only for the flux integral of the curl of the vector field $\iint_S \underline{\underline{\operatorname{curl}\vec{F}}}\cdot\vec{n}\,dS$

Do NOT use it for "regular" surface integrals $\iint_S \underline{\underline{\vec{F}}}\cdot\vec{n}\,dS$

not $\operatorname{curl}\vec{F}$