17.8 The Divergence Theorem (part 1)

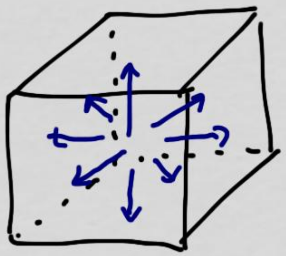
let's take a closer look at the divergence of a vector field

$$\vec{F} = \langle f, g, h \rangle \text{, then } \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle$$

$$= f_x + g_y + h_z$$

what does the divergence tell us?

imagine air is being pumped into a cube (in the center)



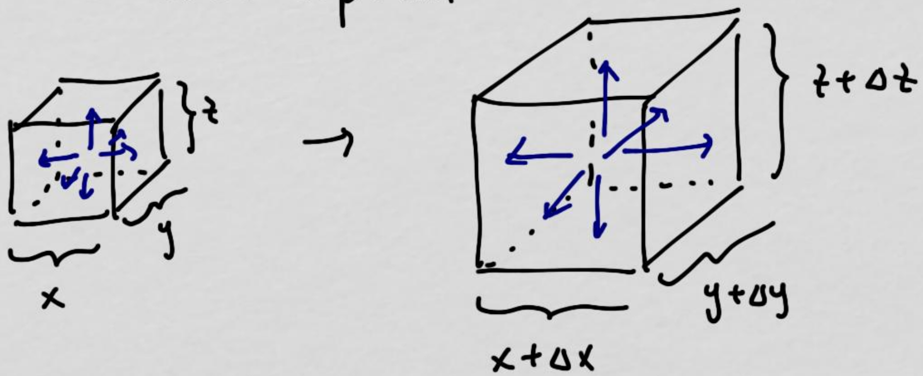
air will create an outward flow from the center and push on all faces of the cube

notice this vector field has a positive divergence (away from a point, so f_x, f_y, f_z are positive)

if $\operatorname{div} \vec{F} > 0 \rightarrow$ flow away from a point

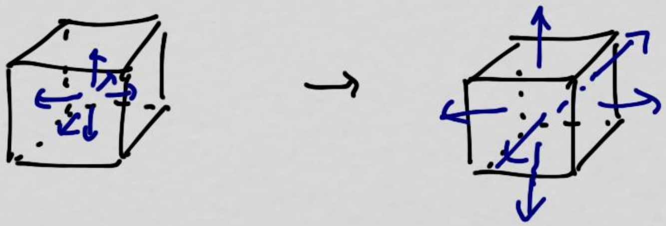
if $\operatorname{div} \vec{F} < 0 \rightarrow$ flow into a point

if the cube is flexible, then the flow will force out the faces
and make the cube expand



if $\text{div}\,\vec{F} > 0$, volume increases

on the other hand, if the faces of the cube are porous, then
instead of the volume getting bigger, the air will flow through
the faces → flux through the surface



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So, clearly, the divergence of a field inside a closed volume is related to the flux through the surface enclosing the volume

→ this is what the Divergence Theorem says

Divergence Theorem

(also known as Gauss' Theorem or Ostrogradsky's Theorem)

$$\underbrace{\iint_S \vec{F}\cdot\vec{n}\,dS}_{\text{flux through surface } S} = \underbrace{\iiint_D \operatorname{div}\vec{F}\,dV}_{\text{accumulation of the divergence inside the volume } D \text{ enclosed by } S}$$

$\vec{n}$: outward normal

for example, for the cube, S consists of all 6 faces and D is the space inside the surface

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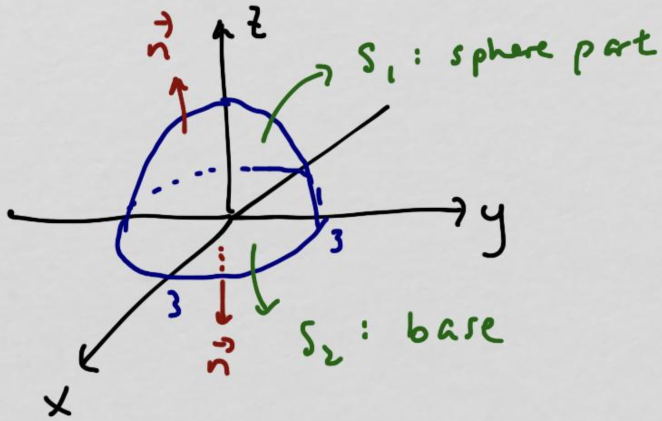
<u>example</u> $\vec{F} = \langle x, y, z \rangle$

S: upper half of a sphere of radius 3, including the circular base at $z = 0$

as usual, $\vec{n}$ is <u>outward</u>



let's verify Divergence Theorem : $\iint\limits_S \vec{F} \cdot \vec{n} \, dS = \iiint\limits_D \operatorname{div} \vec{F} \, dV$

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S_1 : $\vec{r}(u,v) = \langle 3\sin u\cos v,\ 3\sin u \sin v,\ 3\cos u \rangle$ $\quad 0 \le u \le \pi/2 \quad 0 \le v \le 2\pi$

$\phi \qquad \theta$

$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u \rangle$

$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin u \rangle$

is this outward? yes, because z-component ≥ 0, because $0 \le u \le \pi/2$

S_2 : $\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ 0 \rangle$ $\quad 0 \le u \le 3 \quad 0 \le v \le 2\pi$

$r \qquad \theta$

$\vec{r}_u = \langle \cos v,\ \sin v,\ 0 \rangle$

$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0,\ 0,\ u \rangle$ is this outward? no, because z-component is positive with $0 \le u \le 3$, we want it to point <u>down</u>

so, we want $\vec{r}_v \times \vec{r}_u = \langle 0, 0, -u \rangle$ to be our normal on S_2

now do the flux integrals

$$\underbrace{\int_0^{2\pi}\int_0^{\pi/2} \overbrace{\langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle}^{\vec{F}=\langle x,y,z\rangle \text{ using } x,y,z \text{ of } \vec{r}(u,v) \text{ on } S_1} \cdot \overbrace{\langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin u\rangle}^{\vec{r}_u\times\vec{r}_v}\,du\,dv}_{S_1}$$

$$+\underbrace{\int_0^{2\pi}\int_0^{3} \overbrace{\langle u\cos v,\ u\sin v,\ 0\rangle}^{\vec{F}=\langle x,y,z\rangle \text{ using } \vec{r}(u,v) \text{ on } S_2} \cdot \overbrace{\langle 0,\ 0,\ -u\rangle}^{\vec{r}_v\times\vec{r}_u}\,du\,dv}_{S_2}$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (27\sin^3 u + 27\cos^2 u\sin u)\,du\,dv + \int_0^{2\pi}\int_0^{3} 0\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 27\sin u\,du\,dv = \boxed{54\pi}$$

not too hard, but a bit messy

the Divergence Theorem says all that is equal to $\iiint_D \operatorname{div}\vec{F}\,dV$

$\vec{F} = \langle x, y, z \rangle \qquad \operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

so, $\iiint_D \operatorname{div}\vec{F}\,dV = 3 \underbrace{\iiint_D dV}$

volume of the hemisphere with radius 3

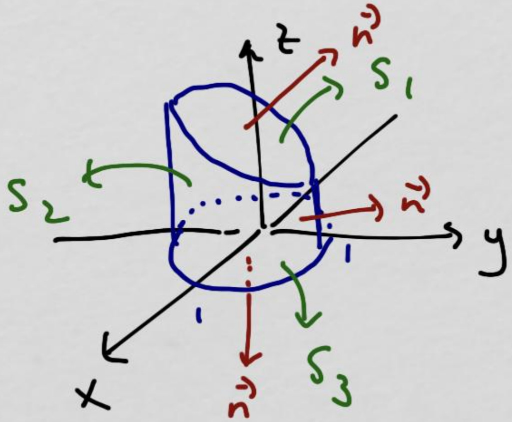
$= \frac{1}{2} \cdot \frac{4}{3}\pi(3)^3$

$= 3 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi(27) = \boxed{54\pi}$ same!

example $\vec{F} = \langle -y^3z^2,\ x^4,\ 4xy^2 \rangle$

S: surface of solid bounded by $\underbrace{x^2+y^2=1}_{\text{cylinder}}$, $\underbrace{z=10-y,\ z=0}_{\text{planes}}$

normal is outward



notice this surface has 3 parts

S_1: plane at top (normal out → z-component > 0)

S_2: cylinder side (normal out → away from z-axis)

S_3: base (normal out → z-component < 0)

looks like messy surface integrals

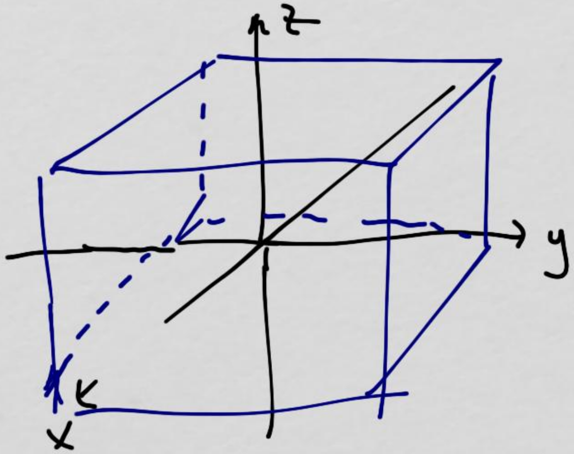
so, the Divergence Theorem is probably very useful

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle -y^3z^2,\ x^4,\ 4xy^2\rangle = \frac{\partial}{\partial x}(-y^3z^2) + \frac{\partial}{\partial y}(x^4) + \frac{\partial}{\partial z}(4xy^2) = 0$$

so, $\iiint_D \operatorname{div}\vec{F}\,dV = 0 = \iint_S \vec{F}\cdot\vec{n}\,dS$ (if we had done the surface integrals, we would get 0, too)

<u>example</u> $\vec{F} = \langle 4y-3x,\ 4z-2y,\ 4y-3x \rangle$

S : faces of the cube $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$



if done as a flux integral, it involves 6 faces, and normal point away from origin

$\operatorname{div} \vec{F} = -3 - 2 + 0 = -5$

simple $\operatorname{div} \vec{F} \rightarrow$ easy volume integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D -5\, dV = -5 \underbrace{\iiint_D dV}_{\substack{\text{volume of cube} \\ \text{sides } 2}}$$

$$= -5(2)^3 = \boxed{-40}$$

negative means the net flow is opposite to the outward normal $\rightarrow$ net flow is <u>into</u> the cube.