14.1 Vector-Valued Functions

scalar-valued functions : $f(t) = \underbrace{t^2+3}$

scalar output

scalar input

$f(1) = 4$

vector-valued functions : $\vec{r}(t) = \underbrace{\langle \cos t, \sin t, 0 \rangle}$

scalar input scalar out

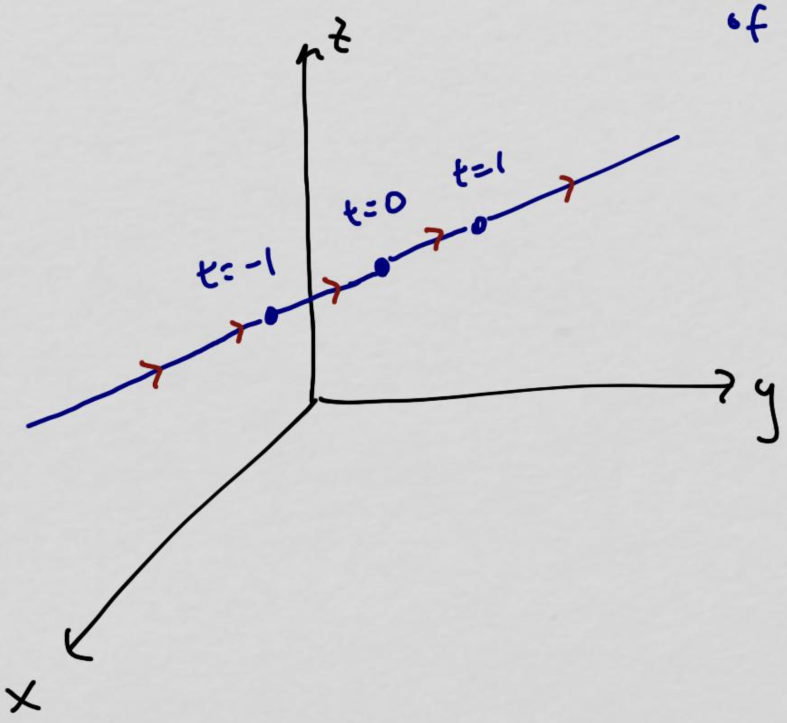
$\vec{r}(0) = \langle 1, 0, 0 \rangle$

example we are familiar with : equation of a line

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

for example, line through $P(1,2,3)$, $Q(4,5,6)$

$$\vec{r}(t) = \underbrace{\langle 1, 2, 3 \rangle}_{\substack{\text{position vector} \\ \text{to } (1,2,3)}} + t \underbrace{\langle 3, 3, 3 \rangle}_{\substack{\text{direction vector} \\ \overrightarrow{PQ}}} = \underbrace{\langle 1+3t,\ 2+3t,\ 3+3t \rangle}_{\substack{\text{position vector to the point} \\ \text{corresponding to the value} \\ \text{of } t}}$$



note the direction we travel as t increases

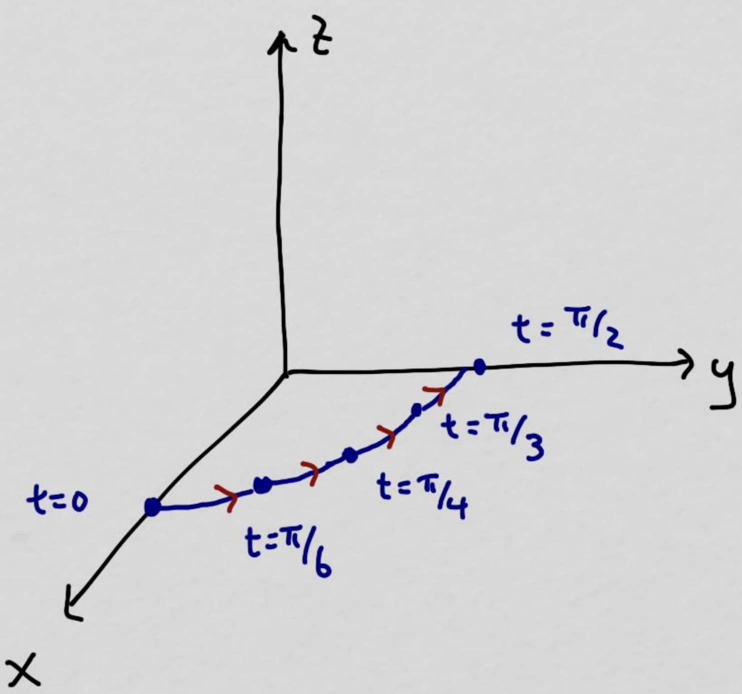
this is called the <u>positive orientation</u>

Draw Erase Select Point Add Undo Redo Rec Stop View Close

example $\vec{r}(t) = \langle \underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{0}_{z(t)} \rangle \quad 0 \le t \le \pi/2$

the simplest way to visualize : plot points and graph

t	x	y	z
0	1	0	0
$\pi/6$	$\sqrt{3}/2$	$1/2$	0
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	0
$\pi/3$	$1/2$	$\sqrt{3}/2$	0
$\pi/2$	0	1	0

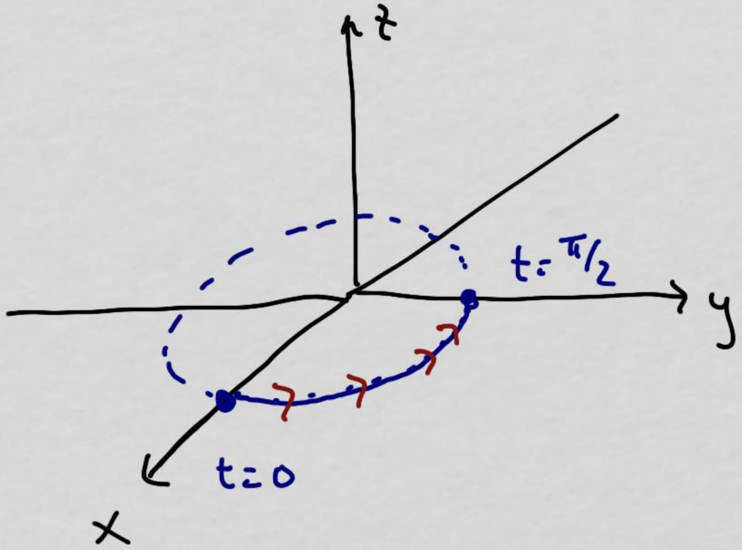


part of a circle, moving counterclockwise when viewed from above another way to visualize : analyze relationship between x, y, z

$$\vec{r}(t) = \langle \underset{x(t)}{\cos t}, \underset{y(t)}{\sin t}, \underset{z(t)}{0} \rangle \qquad 0 \le t \le \pi/2$$

$$\underbrace{x = \cos t, \ y = \sin t}_{x^2 + y^2 = 1}, \ z = 0$$

$x^2 + y^2 = 1$ circle that lies on $z = 0$



but we must not forget $0 \le t \le \pi/2$

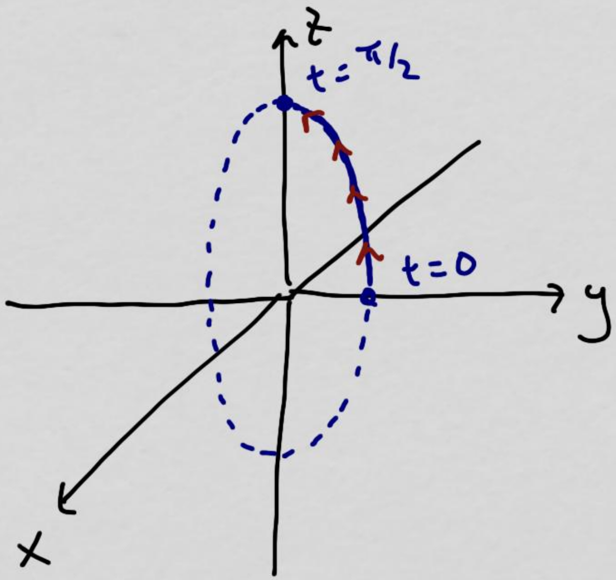
$t = 0$: $x = \cos(0) = 1$
$y = \sin(0) = 0$
$z = 0$

$t = \pi/2$: $x = \cos(\pi/2) = 0$
$y = \sin(\pi/2) = 1$
$z = 0$

if cos t and sin t show up, we are most likely dealing with a circle or ellipse

for example, $\vec{r}(t) = \langle \underset{x(t)}{0}, \underset{y(t)}{\cos t}, \underset{z(t)}{2\sin t} \rangle \qquad 0 \le t \le \pi/2$

$$\left.\begin{aligned} y &= \cos t \\ z &= 2\sin t \end{aligned}\right\} \quad (2y)^2 + z^2 = (2\cos t)^2 + (2\sin t)^2 = 4\cos^2 t + 4\sin^2 t = 4$$



equation: $4y^2 + z^2 = 4$

$$y^2 + \frac{z^2}{4} = 1$$ ellipse

y-ints: ± 1

z-ints: ± 2

$t = 0$: $x = 0, \ y = 1, \ z = 0$

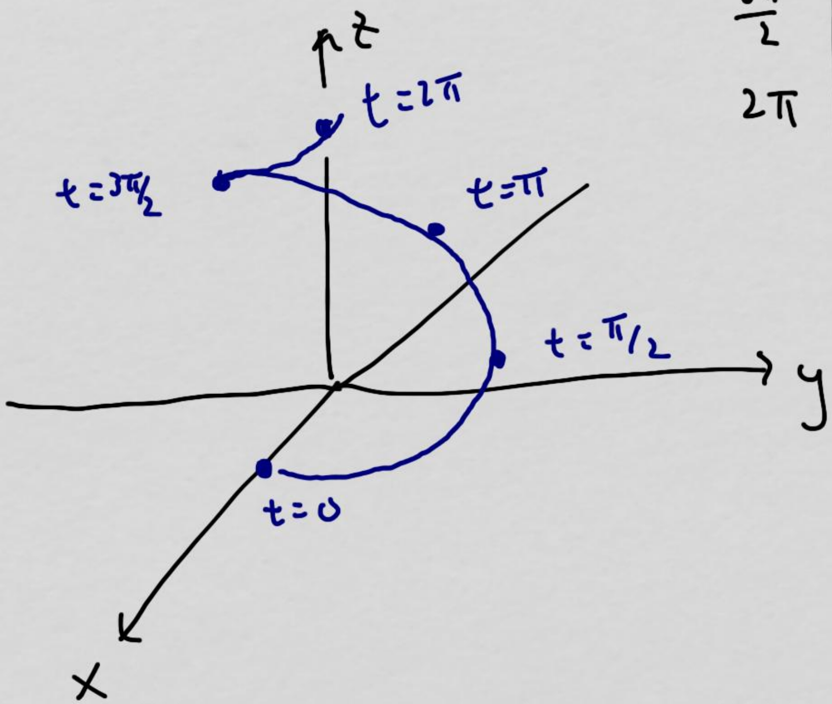
$t = \pi/2$: $x = 0, \ y = 0, \ z = \cancel{x}\ 2$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

<u>example</u> $\vec{r}(t) = \langle \underbrace{\cos t}_{x(t)}, \underbrace{\sin t}_{y(t)}, \underbrace{t}_{z(t)} \rangle \qquad 0 \le t \le 2\pi$

plotting points way:

t	x	y	z
0	1	0	0
$\frac{\pi}{2}$	0	1	$\pi/2$
π	-1	0	π
$\frac{3\pi}{2}$	0	-1	$3\pi/2$
2π	1	0	2π



spiral up (like a spring or a spiral staircase)

cross-sections at $z = k$ are circles

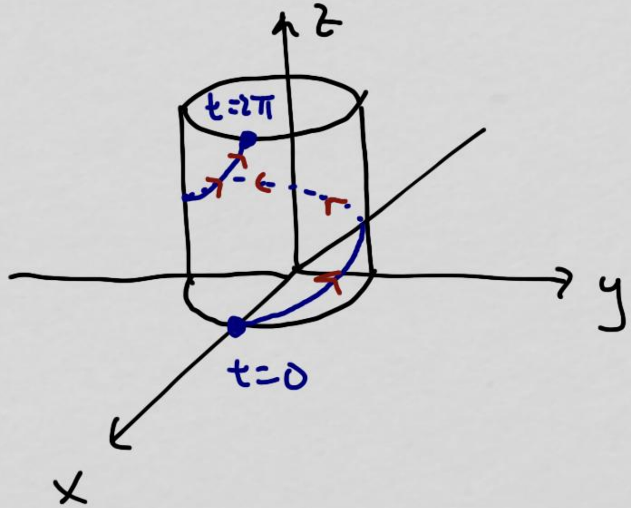
the other way to visualize:

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\} x^2+y^2=1$$

$$z = t$$

shape is $\underbrace{x^2+y^2=1 \quad \text{with} \quad 0 \le z \le 2\pi}$

this describes a cylinder parallel to z-axis
and the curve $\vec{r}(t)$ is on this surface



looking at $\vec{r}(t)$ as a curve on a known surface is usually a more efficient way to visualize (than plotting points)

<u>example</u> $\vec{r}(t) = \langle \underbrace{t\cos t}_{x(t)}, \underbrace{t}_{y(t)}, \underbrace{t\sin t}_{z(t)} \rangle \quad 0 \le t \le 4\pi$

we see $x = t\cos t$, $y = t$, $z = t\sin t$

$$x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = y^2$$

therefore, $\vec{r}(t)$ is part of the surface $\underbrace{x^2 + z^2 = y^2}_{\text{cone}}$

$\vec{r}(t)$ is on this cone starting at $t = 0$ ending at $t = 4\pi$

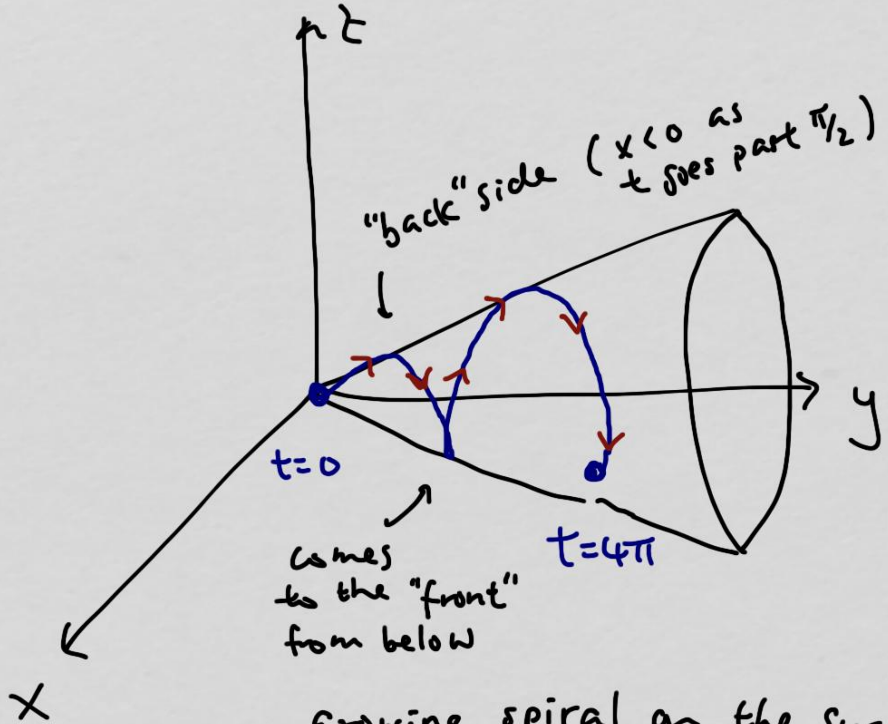
$x^2 + z^2 = y^2$

xz-trace : $x^2 + z^2 = 0$ circle radius 0

as y increases, circle radius increases

xy-trace : $x^2 = y^2$ $y = \pm x$ lines

yz-trace : $y^2 = z^2$ $y = \pm z$ lines



$t = 0$: $x = 0\cos(0) = 0$

$y = 0$

$z = 0\sin(0) = 0$

$t = 4\pi$: $x = 4\pi\cos(4\pi) = 4\pi$

$y = 4\pi$

$z = 4\pi\sin(4\pi) = 0$

fill in in-between points as needed

growing spiral on the surface of cone

<u>example</u> Does the curve $\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle$ $0 \le t \le 2\pi$ intersect the plane $x - z = 0$?

the curve $\vec{r}(t)$ is made up of points $(\underset{x}{t\cos t},\ \underset{y}{t},\ \underset{z}{t\sin t})$ as t goes from 0 to 2π

if $\vec{r}(t)$ intersects the plane $x - z = 0$ at some t, then the point must also be on the plane, <u>so it must satisfy $x - z = 0$</u>

so, we solve $x - z = 0$ using $x = t\cos t$, $y = t$, $z = t\sin t$ for t (the number of solutions = number of intersections)

$x - z = 0 \rightarrow t\cos t - t\sin t = 0$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

$t(\cos t - \sin t) = 0 \qquad 0 \le t \le 2\pi$

$t = 0, \quad \cos t = \sin t$

$t = \pi/4, \ \frac{5\pi}{4}$

the curve $\vec{r}(t) = \langle t\cos t, \ t, \ t\sin t \rangle$ intersects the plane $x - z = 0$ 3 times

at $(0, \ 0, \ 0)$ @ $t = 0$

$\left(\frac{\pi}{4\sqrt{2}}, \ \frac{\pi}{4}, \ \frac{\pi}{4\sqrt{2}}\right)$ @ $t = \pi/4$

$\left(-\frac{5\pi}{4\sqrt{2}}, \ \frac{5\pi}{4}, \ -\frac{5\pi}{4\sqrt{2}}\right)$ @ $t = \frac{5\pi}{4}$

Draw Erase Select Point Add Undo Redo Rec Stop View Close

just like with scalar-valued functions, vector-valued functions

have domains, too

the domain of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the set of all t

where all 3 components $x(t), y(t), z(t)$ are defined.

<u>example</u> $\vec{r}(t) = \left\langle \sqrt{1-t^2}, \sqrt{t}, \frac{1}{\sqrt{5-t}} \right\rangle$

$\sqrt{1-t^2}$ is defined on $[-1, 1]$

$\sqrt{t}$ is defined on $[0, \infty)$

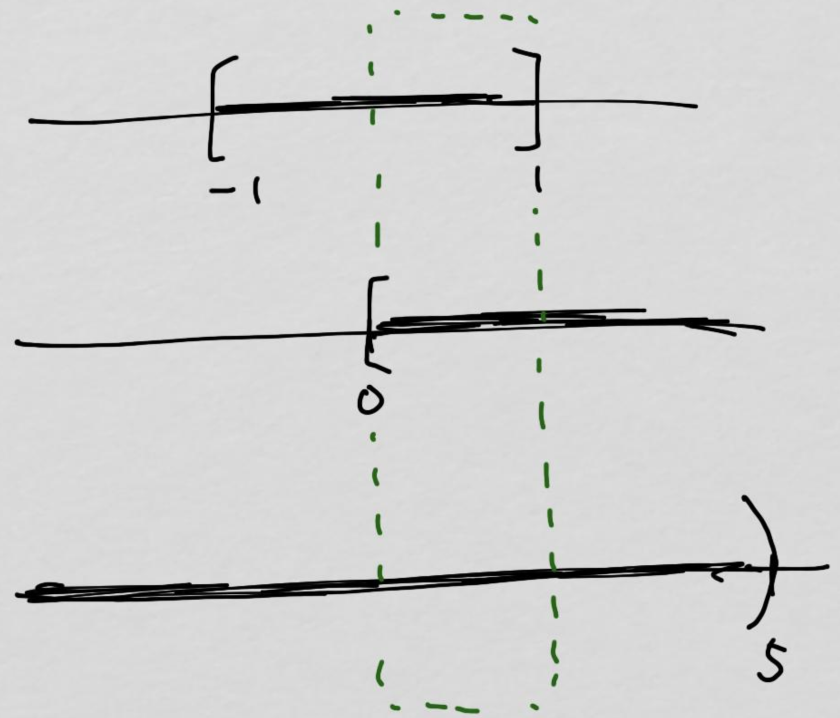
$\frac{1}{\sqrt{5-t}}$ is defined on $(-\infty, 5)$

for all 3 to be defined, we need to find the intersection of these 3 subdomains

$\sqrt{1-t^2}$:

$\sqrt{t}$:

$\dfrac{1}{\sqrt{5-t}}$:



intersection of all 3

domain of $\vec{r}(t)$ is $[0, 1]$