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14.3 Motion in Space (part 2)

if $\vec{r}(t)$ is the position of an object

then $\vec{r}\,'(t) = \vec{v}(t)$ is the velocity

and $\vec{v}\,'(t) = \vec{a}(t)$ is the acceleration

so, if given $\vec{r}(t)$, we can find the acceleration (or force) that is causing that motion

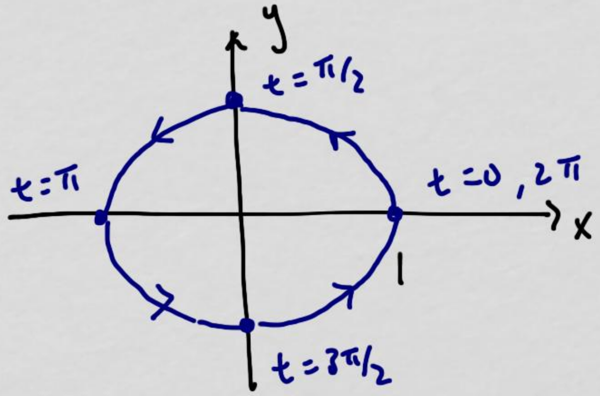
conversely, if given the acceleration (or force), we can integrate to find the resulting motion.

if $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

we know this is a circular motion

because $(x(t))^2 + (y(t))^2 = \cos^2 t + \sin^2 t = 1$

$\rightarrow x^2 + y^2 = 1$ circle radius 1, center at $x=0$, $y=0$



$\vec{r}(t) = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

what kind of acceleration/force gives us this kind of motion

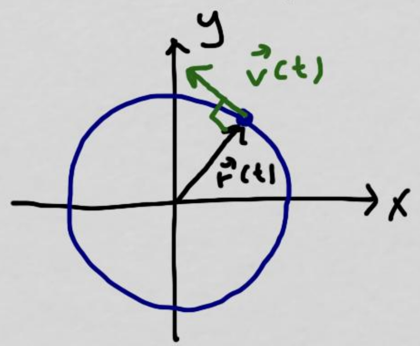
(planets and moons orbit in circular or elliptical paths,
what force causes that?)

$\vec{V}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

by the way, notice $\vec{v} \cdot \vec{r} = \langle -\sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle = 0$

this tells us that $\vec{v}$ is orthogonal to $\vec{r}$
at all t

this is a feature of circular motion



$\vec{v}(t) = \langle -\sin t, \cos t \rangle \qquad 0 \leq t \leq 2\pi$

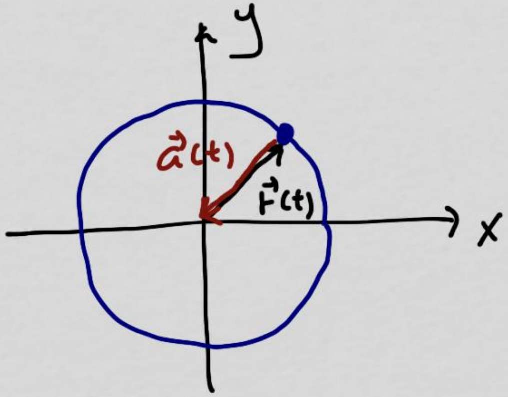
the <u>speed</u> of the object is $|\vec{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

so, the speed is <u>constant</u> while the velocity is not

$\vec{a}(t) = \langle -\cos t, -\sin t \rangle$ this is the acceleration that is causing the motion

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

note $\vec{a}(t) = -\vec{r}(t)$



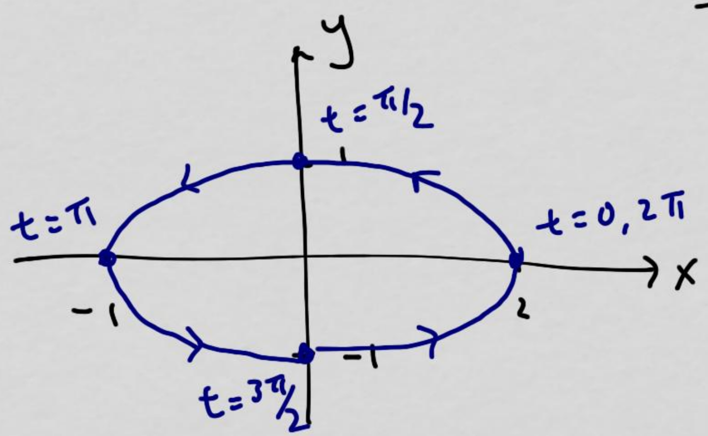
the acceleration points toward the center of the circle

(in the case of, for example, Earth's moon, the Earth's gravity pulls the moon toward the Earth and that force is responsible for the moon going around the Earth) what about an elliptical motion?

$\vec{r}(t) = \langle 2\cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

this path is an ellipse: $\quad x = 2\cos t, \quad y = \sin t$

$$x^2 + (2y)^2 = (2\cos t)^2 + (2\sin t)^2$$
$$= 4\cos^2 t + 4\sin^2 t$$
$$= 4(\cos^2 t + \sin^2 t)$$
$$x^2 + 4y^2 = 4$$
$$\frac{x^2}{4} + y^2 = 1$$



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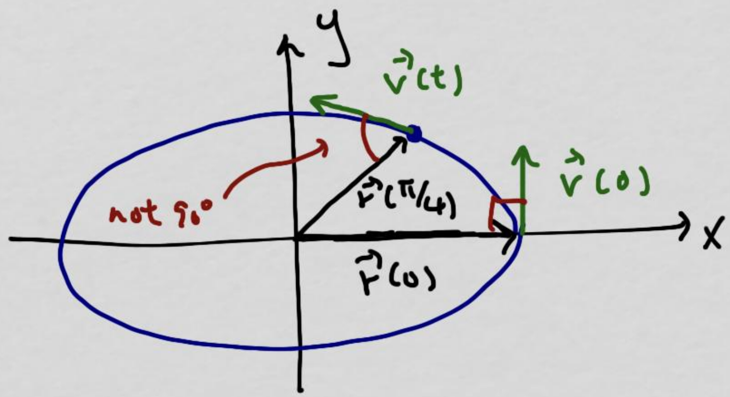
$\vec{r} = \langle 2\cos t, \sin t \rangle$

$\vec{v} = \vec{r}\,' = \langle -2\sin t, \cos t \rangle$

note $\vec{r}$ and $\vec{v}$ are not always orthogonal

$\vec{r} \cdot \vec{v} = -4\cos t \sin t + \sin t \cos t = -3\sin t \cos t \neq 0$ except at

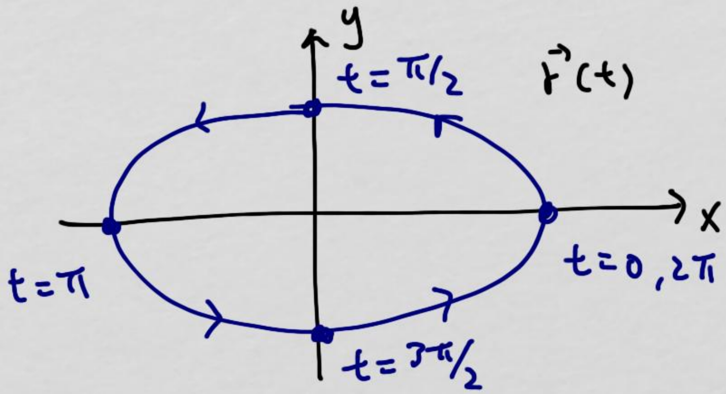
$t = 0, \pi/2, \pi, \frac{3\pi}{2}, 2\pi$

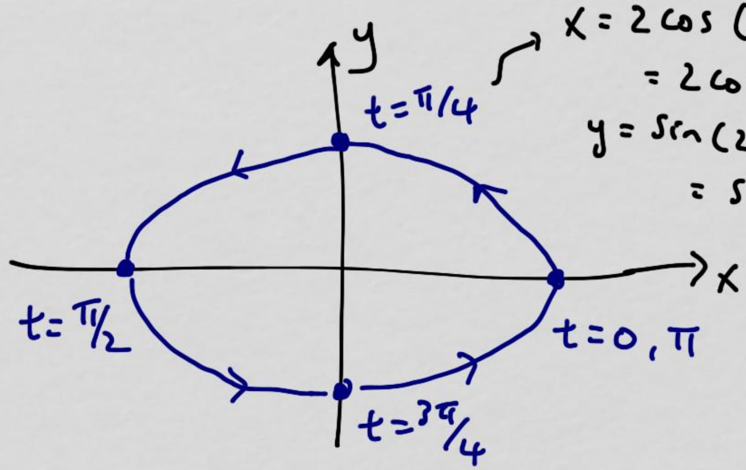


$\vec{a}(t) = \vec{v}\,'(t) = \langle -2\cos t, -\sin t \rangle$

note that it is $-\vec{r}(t)$, so for elliptical motion, the acceleration/force still points toward the center note $\vec{r}(t) = \langle 2\cos t, \sin t \rangle$ and $\vec{R}(t) = \langle 2\cos(\underline{2}t), \sin(\underline{2}t) \rangle$

trace out the same path





$x = 2\cos(2 \cdot \pi/4)$
$= 2\cos(\pi/2) = 0$
$y = \sin(2 \cdot \pi/4)$
$= \sin(\pi/2) = 1$

Same path but it takes an object on the trajectory described by $\vec{R}(t)$ half as much time as an object on the trajectory described by $\vec{r}(t)$ to complete one cycle

$t = 1 \cdot t$

$\vec{r}(t) = \langle 2\cos t, \sin t \rangle \rightarrow$ period $= \dfrac{2\pi}{1} = 2\pi$

$\vec{R}(t) = \langle 2\cos(2t), \sin(2t) \rangle \rightarrow$ period $= \dfrac{2\pi}{2} = \pi$

if it takes less time on $\vec{R}(t)$, the speed of the object on $\vec{R}$ must be bigger

$$\vec{r}(t) = \langle 2\cos t, \sin t \rangle$$

$$\vec{R}(t) = \langle 2\cos(2t), \sin(2t) \rangle$$

$$\vec{V}_r(t) = \langle -2\sin t, \cos t \rangle$$

$$\vec{V}_R(t) = \langle -4\sin(2t), 2\cos(2t) \rangle$$

$$|\vec{v}_r(t)| = \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{3\sin^2 t + \underbrace{\sin^2 t + \cos^2 t}_{1}} = \sqrt{1 + 3\sin^2 t}$$

$$|\vec{V}_R(t)| = \sqrt{16\sin^2(2t) + 4\cos^2(2t)} = \sqrt{12\sin^2(2t) + \underbrace{4\sin^2(2t) + 4\cos^2(2t)}_{4}}$$

$$= \sqrt{4 + 12\sin^2(2t)}$$

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we know $-1 \leq \sin t \leq 1$, so $0 \leq \sin^2(t) \leq 1$

similarly, $0 \leq \sin^2(2t) \leq 1$

$|\vec{v}_r| = \sqrt{1 + 3\sin^2 t} \longrightarrow 1 \leq |\vec{v}_r| \leq 2$

$|\vec{v}_R| = \sqrt{4 + 12\sin^2(2t)} \longrightarrow 2 \leq |\vec{v}_R| \leq 4$

so, the speed of the object on $\vec{R}$ is greater (which makes sense because the cycle is completed faster)

now let's turn the analysis around, start w/ a known acceleration (or force) and then find the resulting trajectory

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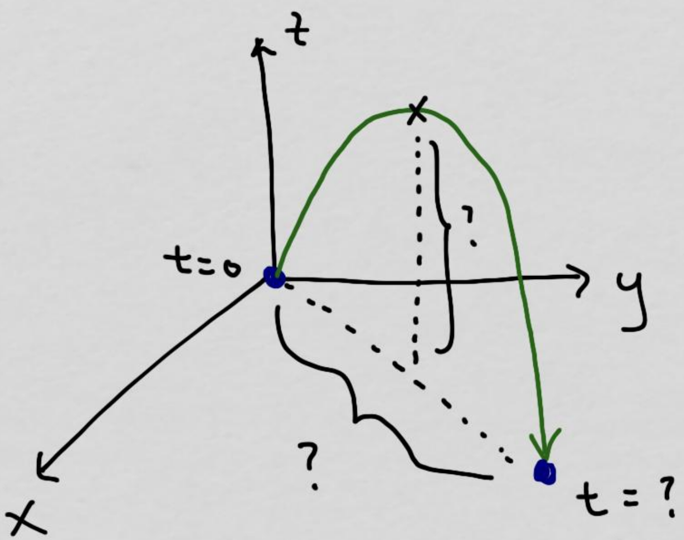
<u>example</u> A ball resting on the ground is kicked with an initial velocity of $\vec{v}(0) = \langle 10, 15, 20 \rangle$ m/s

If the only acceleration acting on the ball is gravity, how long will the ball stay in the air? How far does it fly? What is its maximum altitude?



assume the original position is at the origin

$\vec{r}(0) = \langle 0, 0, 0 \rangle$

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we are told that the only acceleration is due to gravity

so, $\vec{a}(t) = \langle 0, 0, -9.8 \rangle$ m/s2

integrate to find velocity:

$\vec{V}(t) = \int \vec{a}(t)\,dt = \langle C_1, C_2, -9.8t + C_3 \rangle$

now use the given $\vec{v}(0) = \langle 10, 15, 20 \rangle$ to find C_1, C_2, C_3

$\vec{v}(0) = \langle C_1, C_2, 0 + C_3 \rangle = \langle 10, 15, 20 \rangle$

so, $C_1 = 10$, $C_2 = 15$, $C_3 = 20$

therefore, $\boxed{\vec{v}(t) = \langle 10, 15, -9.8t + 20 \rangle}$

integrate again to find position

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 10t + d_1, 15t + d_2, -4.9t^2 + 20t + d_3 \rangle$

now use the given $\vec{r}(0) = \langle 0, 0, 0 \rangle$ to find d_1, d_2, d_3

$$\vec{r}(t) = \langle 10t + d_1,\ 15t + d_2,\ -4.9t^2 + 20t + d_3 \rangle$$

$$\vec{r}(0) = \langle 0 + d_1,\ 0 + d_2,\ 0 + 0 + d_3 \rangle = \langle 0, 0, 0 \rangle$$

so, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$

therefore, $\boxed{\vec{r}(t) = \langle 10t,\ 15t,\ -4.9t^2 + 20t \rangle}$

How long does it fly (time of flight)?

the z-componet of $\vec{r}(t)$ tells us the height

$$z = -4.9t^2 + 20t$$

let's find t when $z = 0 \rightarrow$ we expect two : start of the flight
end of the flight

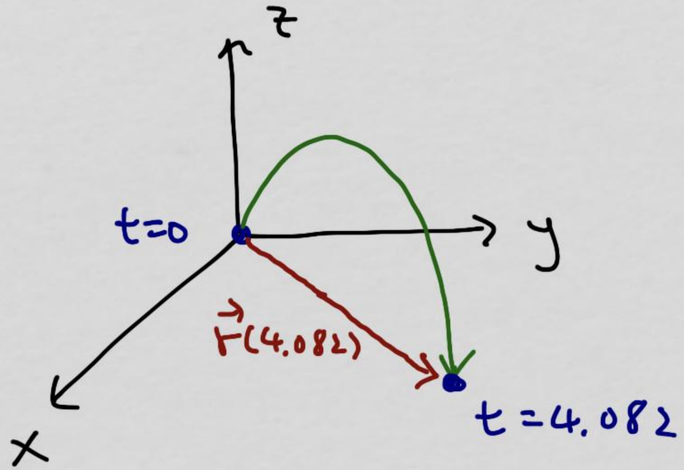
$$0 = -4.9t^2 + 20t \longrightarrow 0 = t(-4.9t + 20)$$

$t = 0$, $\quad t = \dfrac{20}{4.9} = 4.082$

start $\quad\quad$ end

so, $\boxed{\text{it stays in the air for } 4.082 \text{ seconds}}$

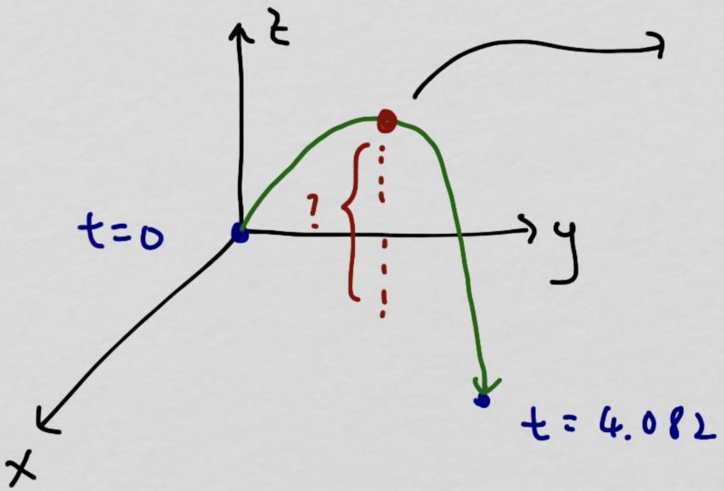
what is its range (distance from the start to the point where it lands)?



so, the range is simply $|\vec{r}(4.082)|$

$\vec{r}(4.082) = \langle 40.82, 61.23, 0 \rangle$

$\boxed{|\vec{r}(4.082)| = 73.59}$ range is 73.59 meters finally, the maximum height



at the point of max height,
the z-component of velocity
is momentarily zero
(it's not going up or down)

$$\vec{v}(t) = \langle 10, 15, \underbrace{-9.8t + 20}_{\text{when is this zero?}} \rangle$$

$$t = \frac{20}{9.8} = 2.041$$

so, at $t = 2.041$, the vertical component of velocity is zero,
so maximum height is attained at that time

max height: vertical (z) component of $\vec{r}$

from $\vec{r}(t)$, $z = -4.9t^2 + 20t = -4.9(2.041)^2 + 20(2.041) = 20.409$

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$\boxed{\text{So, the max height is } 20.409 \text{ meters}}$

follow-up question: what happens if we double the initial velocity? does it fly twice as far?

repeat the process, but with $\vec{v}(0) = 2\langle 10, 15, 20\rangle = \underline{\underline{\langle 20, 30, 40\rangle}}$

$\vec{a}(t) = \langle 0, 0, -9.8\rangle$

$\vec{v}(t) = \langle C_1, C_2, -9.8t + C_3\rangle$

$\vec{v}(0) = \langle C_1, C_2, C_3\rangle = \langle 20, 30, 40\rangle$

so, $\boxed{\vec{v}(t) = \langle 20, 30, -9.8t + 40\rangle}$

$\vec{r}(t) = \langle 20t + d_1, 30t + d_2, -4.9t^2 + 40t + d_3\rangle$

$\vec{r}(0) = \langle d_1, d_2, d_3\rangle = \langle 0, 0, 0\rangle$

So, $\boxed{\vec{r}(t) = \langle 20t,\ 30t,\ -4.9t^2 + 40t \rangle}$

time of flight: $z = -4.9t^2 + 40t = 0$

$t(-4.9t + 40) = 0 \to t = 0, \quad t = \frac{40}{4.9} = 8.164$

original time of flight : 4.082

> by doubling the initial velocity, we double the time of flight

what about the range?

$$|\vec{r}(8.164)| = \sqrt{(20(8.164))^2 + (30(8.164))^2 + (-4.9(8.164)^2 + 40(8.164))^2}$$

$$= 294.357$$

original : 73.59

> by doubling the initial velocity, we quadruple the range