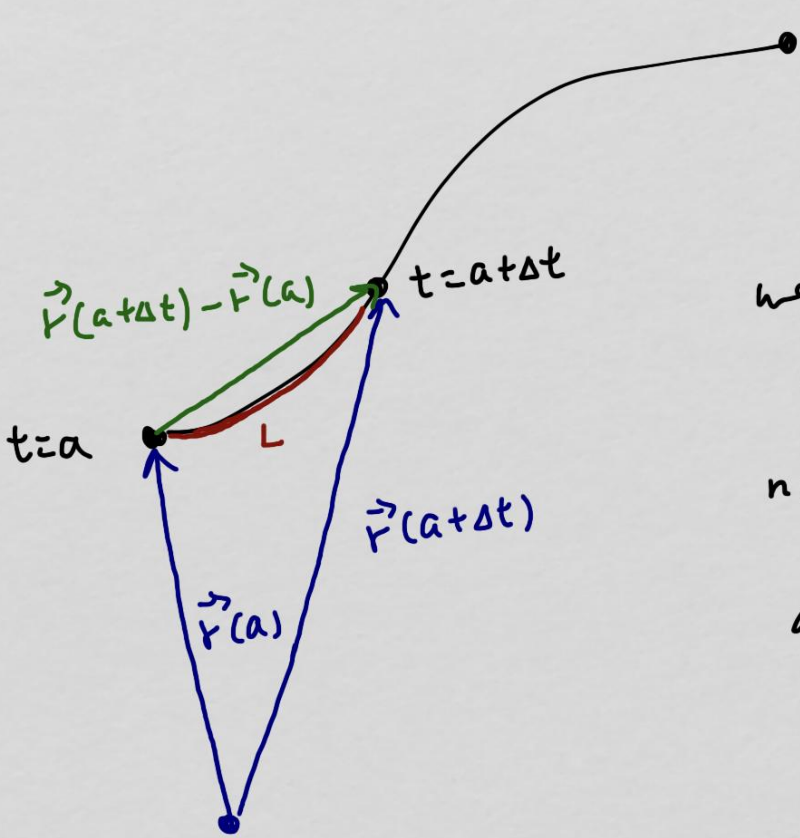
14.4 Length of Curves

$\vec{r}(t)$, $a \le t \le b$



How long is this curve?

we want to find L : length of curve from $t=a$ to $t=a+\Delta t$

notice that $L \approx \vec{r}(a+\Delta t) - \vec{r}(a)$

and the approximation gets better if Δt is small from last time, $\vec{r}\,'(a) = \lim\limits_{h \to 0} \dfrac{\vec{r}(a+h) - \vec{r}(a)}{h}$

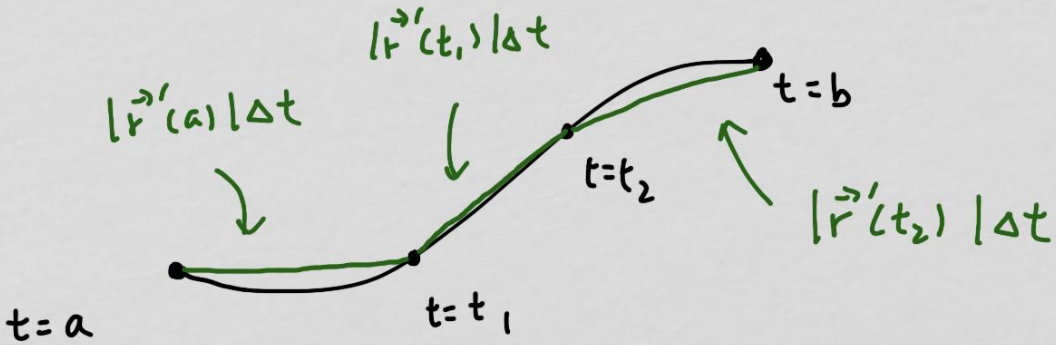
so, $\vec{r}\,'(a) \approx \dfrac{\vec{r}(a+h) - \vec{r}(a)}{h} \approx \dfrac{\vec{r}(a+\Delta t) - \vec{r}(a)}{\Delta t}$

therefore, $\vec{r}\,'(a)\,\Delta t \approx \underbrace{\vec{r}(a+\Delta t) - \vec{r}(a)}_{\text{the green vector on last page, which we use to estimate } L}$

this means, the length of the curve from $t=a$ to $t=a+\Delta t$

is roughly $\left|\vec{r}(a+\Delta t) - \vec{r}(a)\right| \approx \left|\vec{r}\,'(a)\right| \Delta t$ (assume $\Delta t > 0$)

so, the length of curve from $t=a$ to $t=a+\Delta t$ is approximately equal to $\left|\vec{r}\,'(a)\right| \Delta t$



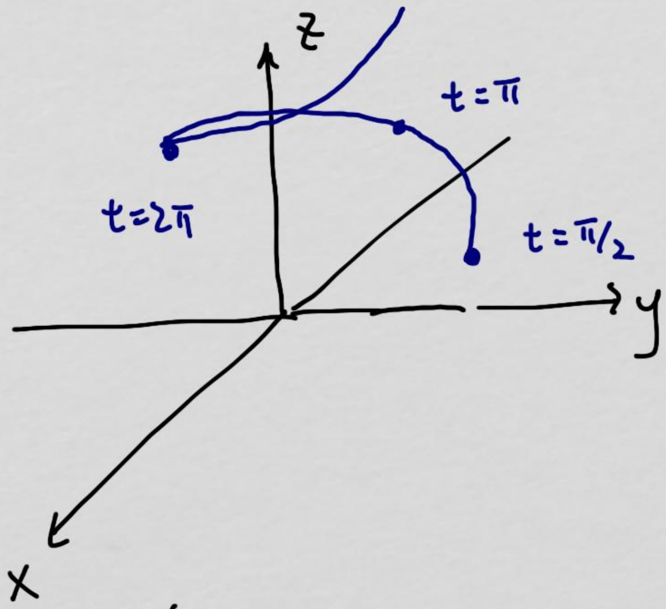
total length $\approx |\vec{r}\,'(a)|\Delta t + |\vec{r}\,'(t_1)|\Delta t + |\vec{r}\,'(t_2)|\Delta t$

but we want the <u>exact</u> length, so we shrink the intervals to be really small : $\Delta t \to dt$ and the sum becomes an infinite sum which becomes an integral

therefore, the total length from $t=a$ to $t=b$ is

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

<u>example</u> $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad \frac{\pi}{2} \le t \le 2\pi$



$t = \frac{\pi}{2}: \quad \vec{r}\left(\frac{\pi}{2}\right) = \langle 0, 1, \frac{\pi}{2} \rangle$

$t = \pi: \quad \vec{r}(\pi) = \langle -1, 0, \pi \rangle$

this is a helix (spiral on surface of cylinder $x^2+y^2=1$)

$$L = \int_a^b |\vec{r}\,'(t)|\, dt$$

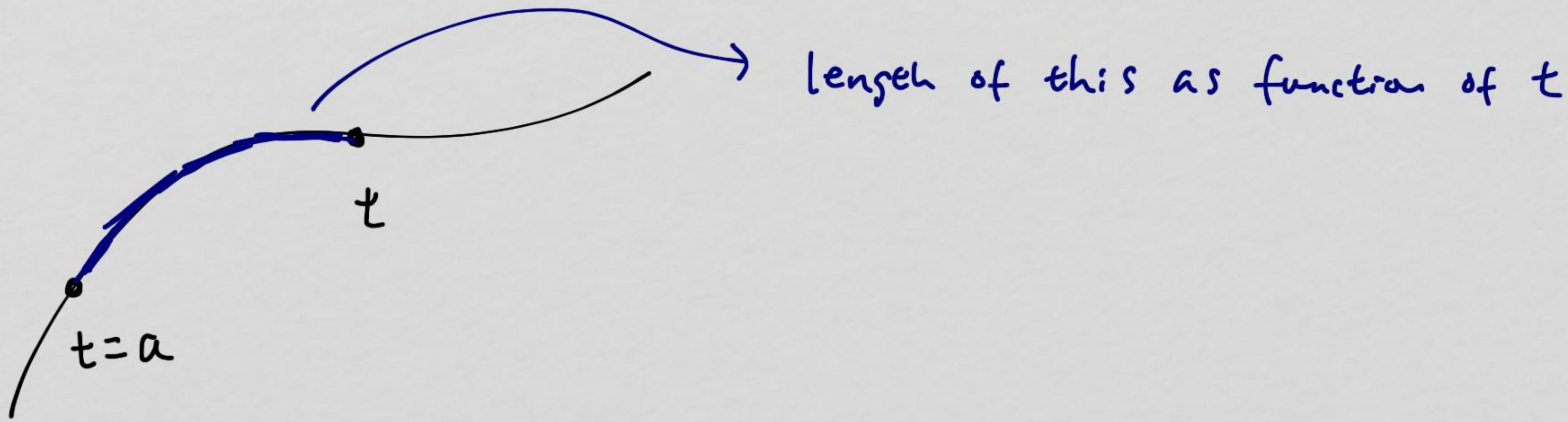
$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$L = \int_{\frac{\pi}{2}}^{2\pi} \sqrt{2}\, dt = \sqrt{2}\left(2\pi - \frac{\pi}{2}\right) = \sqrt{2}\left(\frac{3\pi}{2}\right) = \boxed{\frac{3\pi}{\sqrt{2}}}$$

exact length

that formula can be tweaked a bit to find the length from $t=a$ to some undetermined t $\rightarrow$ length as a function of t

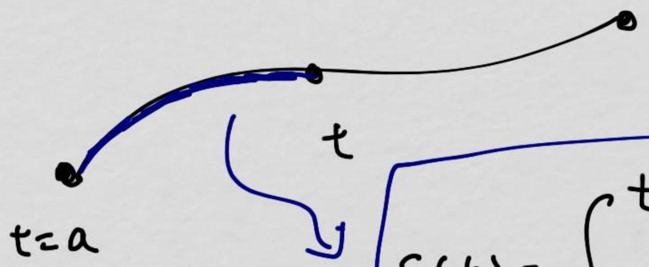


$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

now instead of using a fixed value b as the upper bound, let's let it be t (but we don't want to use the same symbol in our integral)

$$= \int_a^t |\vec{r}\,'(u)|\,du$$

u: "dummy variable" (we can't have the same variable in limit as in integral)



$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du}$$

arc length from $t=a$ to t

try it out on the helix: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

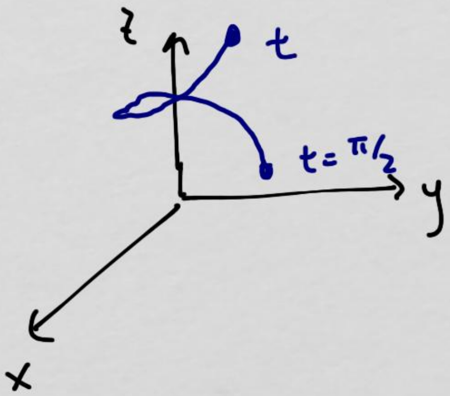
$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

length of curve from $t = \pi/2$ to t: $\quad s(t) = \displaystyle\int_{\frac{\pi}{2}}^{t} |\vec{r}\,'(u)|\,du$

$$s(t) = \int_{\frac{\pi}{2}}^{t} \sqrt{2}\, du = \sqrt{2}\, u \Big|_{\frac{\pi}{2}}^{t} = \boxed{\sqrt{2}\left(t - \frac{\pi}{2}\right)}$$

length of $\vec{r}(t)$ from $t = \pi/2$ to whatever t



now, if we let $t = 2\pi$, this formula should give the same length as when we found length from $t = \pi/2$ to $t = 2\pi$ using $L = \int_a^b |\vec{r}\,'(t)|\, dt$ (which was $\frac{3\pi}{\sqrt{2}}$)

check: $s(t) = \sqrt{2}\left(t - \frac{\pi}{2}\right)$

$$s(2\pi) = \sqrt{2}\left(2\pi - \frac{\pi}{2}\right) = \sqrt{2}\left(\frac{3\pi}{2}\right) = \frac{3\pi}{\sqrt{2}}$$

so it matches, as it should

also, if $t = \pi/2$, then length should be zero $\quad s(\pi/2) = \sqrt{2}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0$

$\underbrace{\vec{r}(t) = \langle \cos t, \sin t, t \rangle}_{\text{gives us the location at a time we specify}}$ is the curve using t as the parameter
(t is often, but not always, time)

we can actually re-parametrize the curve using arc length s as the parameter

$\vec{r}(s) \rightarrow$ gives us the location after a length of s has been traversed.

try this on the helix example again

<u>example</u> $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \quad \frac{\pi}{2} \leq t \leq 2\pi$

rewrite this using arc length as the parameter

from earlier, we found $s(t) = \sqrt{2}\,(t - \frac{\pi}{2})$

so,
$$s = \sqrt{2}\,(t - \tfrac{\pi}{2})$$
$$\frac{s}{\sqrt{2}} = t - \frac{\pi}{2}$$
$$t = \frac{s}{\sqrt{2}} + \frac{\pi}{2} = \frac{s}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} + \frac{\pi}{2} = \frac{\sqrt{2}\,s}{2} + \frac{\pi}{2} = \frac{\sqrt{2}\,s + \pi}{2}$$

now, we go back to $\vec{r}(t)$ and sub out t using

$$\vec{r}(s) = \left\langle \cos\left(\frac{\sqrt{2}\,s+\pi}{2}\right), \sin\left(\frac{\sqrt{2}\,s+\pi}{2}\right), \frac{\sqrt{2}\,s+\pi}{2} \right\rangle$$

range of s ?

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$$\frac{\pi}{2} \le t \le 2\pi \qquad \text{and} \qquad s = \sqrt{2}\left(t - \frac{\pi}{2}\right)$$

$$0 \le s \le \frac{3\pi}{\sqrt{2}} \longrightarrow s(2\pi) = \sqrt{2}\left(2\pi - \frac{\pi}{2}\right)$$

$$s\left(\frac{\pi}{2}\right) = \sqrt{2}\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = 0$$

So, the same curve, using arc length as the parameter is

$$\boxed{\begin{gathered}\vec{r}(s) = \left\langle \cos\left(\frac{\sqrt{2}s+\pi}{2}\right), \sin\left(\frac{\sqrt{2}s+\pi}{2}\right), \frac{\sqrt{2}s+\pi}{2} \right\rangle \\ 0 \le s \le \frac{3\pi}{\sqrt{2}}\end{gathered}}$$

$$\vec{r}(1) = \left\langle \cos\left(\frac{\sqrt{2}+\pi}{2}\right), \sin\left(\frac{\sqrt{2}+\pi}{2}\right), \frac{\sqrt{2}+\pi}{2} \right\rangle$$ is where we are after we traveled a length of 1 from the starting point

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if we plug in $t=1$ into $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ it gives us

where we are at $t=1$ (time is 1 second, for example)

it's easy to tell whether we are using time (t) or length (s)

as our parameter if we always use t for time and s for length.

However, they are just variables and we can use whatever we want for whatever we want

→ we can use t for length if we want.

if we see $\vec{r}(t)$, how do we know that t is not s in disguise?

for example, in $\vec{r}(t) = \langle \cos t, 0, \sin t \rangle$ $\quad 0 \le t \le 3$

is this t really time or is it length in disguise?

recall $\quad s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

differentiate both sides with respect to t

$$\frac{d}{dt} s(t) = \frac{d}{dt}\int_a^t |\vec{r}\,'(u)|\,du$$

$$= |\vec{r}\,'(t)| \quad \text{(from the Fundamental Theorem of Calculus)}$$

so, $\quad \dfrac{ds}{dt} = |\vec{r}\,'(t)|$

if t is really s, then $t = s$, so $\frac{ds}{dt} = 1$

this means, since $\frac{ds}{dt} = |\vec{r}\,'(t)|$, if $t = s$, then $\boxed{|\vec{r}\,'(t)| = 1}$

so, if this t is really the length s, then $|\vec{r}\,'(t)| = 1$

$\vec{r}(t) = \langle \cos t, 0, \sin t \rangle \qquad 0 \le t \le 3$

$\vec{r}\,'(t) = \langle -\sin t, 0, \cos t \rangle$

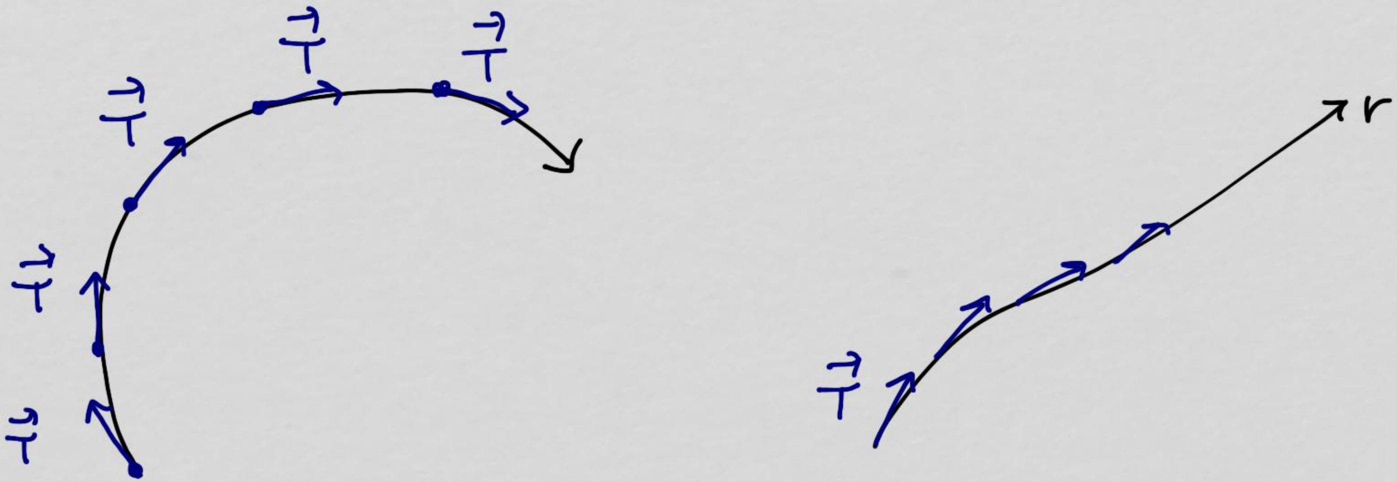
$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

therefore, the t in this equation is really the arc length s

14.5 Curvature

recall the unit tangent vector $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

since the length of $\vec{T}$ is always 1 (unit vector), whatever rate of change $\vec{T}$ has must be due to the change in direction



note that $\vec{T}$ changes direction much more on a curvy curve than on a flat one therefore, the rate of change of $\vec{T}$ as we move along the curve must be much greater

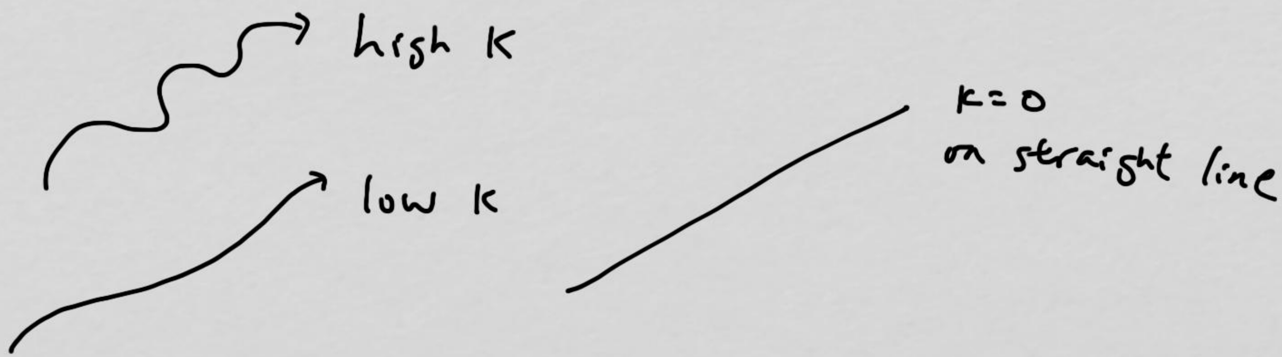
so, $\left|\dfrac{d\vec{T}}{ds}\right|$ is greater on curvy curve than on flat curve

arc length

we call this quantity <u>the curvature</u> and give it the symbol κ (Greek letter "kappa")

$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right|}$$

curvature



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that form is not always convenient, because we have to involve arc length s

we can rewrite it to make using it a little easier:

$$K = \left| \frac{d\vec{T}}{ds} \right| = \frac{|d\vec{T}|}{|ds|} = \frac{|d\vec{T}/dt|}{|ds/dt|}$$

↳ from last section

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

so, K is also

$$K = \frac{\left|\frac{d\vec{T}(t)}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

$$\boxed{K = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}}$$

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<u>example</u>

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{2}}$$

$$= \left\langle -\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{T}\,' = \left\langle -\frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{2}} \sin t, 0 \right\rangle$$

$$|\vec{T}\,'| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

so, $$K = \frac{|\vec{T}\,'|}{|\vec{r}\,'(t)|} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \boxed{\frac{1}{2}}$$

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there are other alternative forms of k

one of which you will see on homework is

$$K = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3}$$ or if $\vec{r}\,' = \vec{v}$ and $\vec{r}\,'' = \vec{a}$

$$K = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}$$