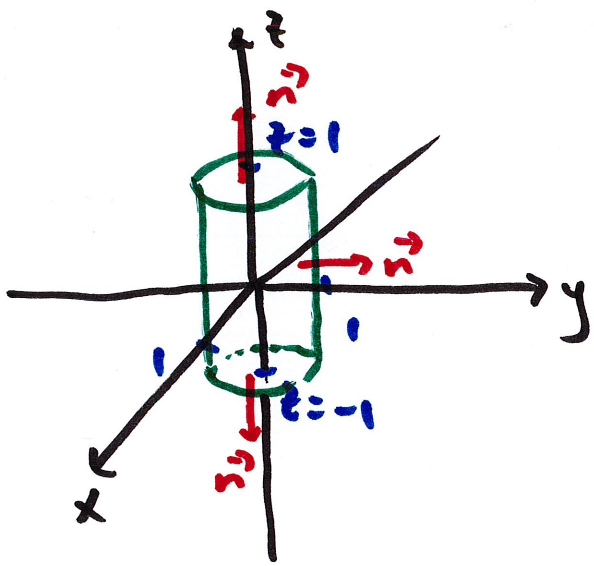
Suppose $\vec{F}(x,y,z) = 2xy^2\,\vec{i} + 2yx^2\,\vec{j} - (x^2+y^2)z\,\vec{k}$

and $S$ is boundary of solid enclosed by cylinder $x^2+y^2=1$ and planes $z=-1$ and $z=1$.

$S$ is a closed surface oriented by outward normal.

Calculate flux integral $\iint_S \vec{F}\cdot\vec{n}\,dS$




Options: 3 surface integrals (top, bottom, side)

Divergence Theorem

why? closed surface flux integral

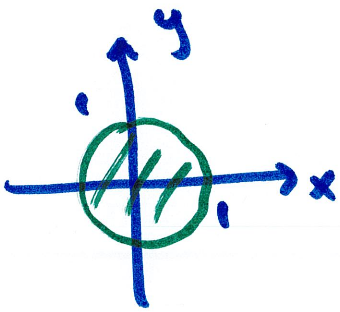
seems better here

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}\vec{F}\,dV$$

↳ enclosed space

$$\operatorname{div}\vec{F} = 2y^2 + 2x^2 - (x^2+y^2) = x^2+y^2$$

the enclosed space is inside cylinder $\rightarrow$ use cylindrical



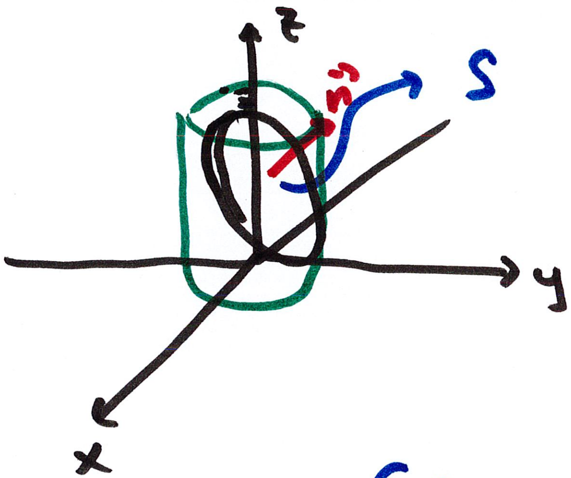

$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$
$$-1 \le z \le 1$$

$$\operatorname{div}\vec{F} = x^2+y^2 = r^2$$

$$\iiint_E \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^1\int_{-1}^1 r^2 \, r\, dz\, dr\, d\theta = \dots = \cancel{4}\pi$$

Let $S$ be the part of the plane $y+z=10$ that lies inside the cylinder $x^2+y^2=1$.

Compute $\iint_S \vec{F}\cdot\vec{n}\,dS$ for $\vec{F}(x,y,z) = \langle x,\ 1-y+e^z,\ y-e^z \rangle$ with $S$ oriented by upward normal.



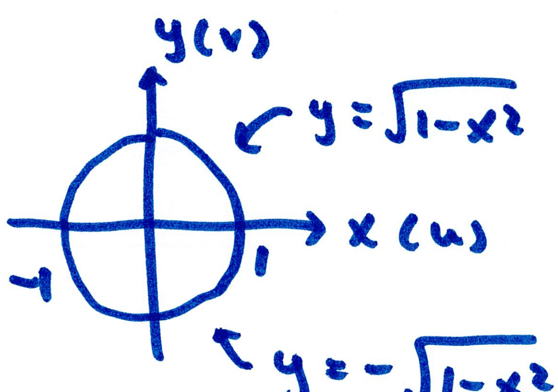

calculate $\iint_S \vec{F}\cdot\vec{n}\,dS$

only one so shouldn't be too bad

parameterize $S$

Cartesian: $\vec{r}(u,v) = \langle u,\ v,\ 10-v \rangle$ ← from $y+z=10$

x ↗ (u), ↖ y (v)




$$-1 \le u \le 1$$

$$-\sqrt{1-u^2} \le v \le \sqrt{1-u^2}$$

$\vec{r}_u = \langle 1,\ 0,\ 0 \rangle$

$\vec{r}_v = \langle 0,\ 1,\ -1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0,\ 1,\ 1 \rangle$ upward? yes.

$$\vec{F} = \langle x,\ 1 - y + e^{z},\ y - e^{z} \rangle$$

$$= \langle u,\ 1 - v + e^{10-v},\ v - e^{10-v} \rangle$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_R \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA$$

$$= \iint_R \left(1 - v + e^{10-v} + v - e^{10-v}\right) dA$$

$$= \underbrace{\int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv\, du}_{\text{area of circle radius } 1} = \pi$$

Consider $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ where $\vec{r} = \langle x, y, z \rangle$

and $|\vec{r}| = (x^2+y^2+z^2)^{1/2}$ which of the following is/are true:

i) $\int_C \vec{F} \cdot d\vec{r}$ is independent of path

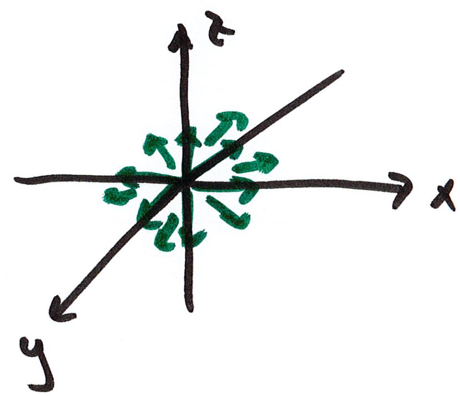
ii) $\iint_S \vec{F} \cdot \vec{n}\, dS = 0$ for any closed surface $S$ that encloses the origin

iii) $\text{div}(\vec{F}) = 0$

i) $\int_C \vec{F} \cdot d\vec{r}$ is indp of path if $\vec{F}$ is conservative

$\text{curl}\,\vec{F} = \vec{0}$

or no rotation in the $\vec{F}$




no rotation

so $\vec{F}$ is

Conservative

so i) is TRUE

iii) $\operatorname{div} \vec{F} = 0$

$$\vec{F} = \left\langle \underbrace{\frac{x}{(x^2+y^2+z^2)^{3/2}}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\frac{\partial}{\partial x}\left( \quad \right) = \frac{(x^2+y^2+z^2)^{3/2}(1) - (x)\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}\left[(x^2+y^2+z^2) - 3x^2\right]}{(x^2+y^2+z^2)^3}$$

$$= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial y}(\text{middle one}) = \dots = \frac{x^2+y^2+z^2-3y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial z}(\text{last one}) = \dots = \frac{x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\operatorname{div} \vec{F} = \frac{0}{(x^2+y^2+z^2)^{5/2}}$$

$$= 0$$

iii) is TRUE

ii) $\iint_S \vec{F}\cdot\vec{n}\,dS = 0$ for any $S$ that encloses origin

$S$ is closed

& Div. Theorem: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}\vec{F}\,dV$

appears to be true since $\operatorname{div}\vec{F} = 0$

BuT, $\vec{F}$ is not defined at origin,

so we cannot apply Div. Theorem

so we cannot say the flux is 0.

ii) is FALSE

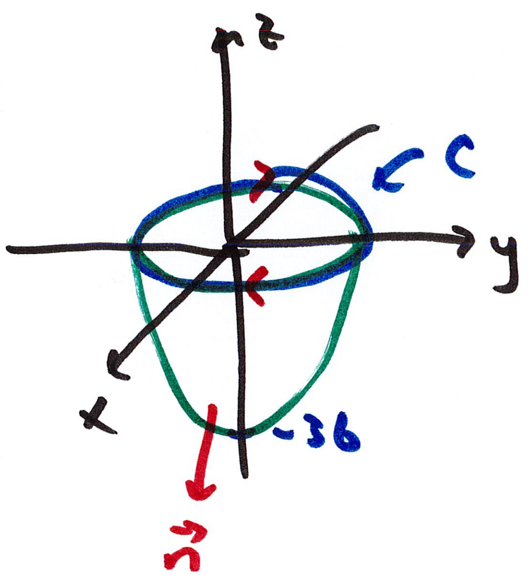
Let $\vec{F} = (y + z\cos x)\,\vec{i} + (-x + z\sin y)\,\vec{j} + (xye^{z})\,\vec{k}$

compute $\displaystyle\iint_S \underbrace{(\vec{\nabla}\times\vec{F})}_{\text{curl }\vec{F}}\cdot\vec{n}\,dS$

↳ curl $\vec{F}$ Stokes' might help

$S$ : part of $z = f(x,y) = e^{x}(x^2+y^2-36)$ below $xy$-plane

with downward normal.

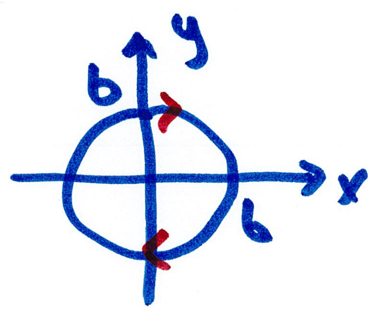



Stokes' : $\displaystyle\iint_S \text{curl }\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

$C$ : $z = \underbrace{e^{x}}_{\neq 0}\underbrace{(x^2+y^2-36)}_{=0} = 0$

so $C : x^2+y^2 = 36$

circle radius 6




$\vec{r}(t) = \langle 6\cos t,\ 6\sin t,\ 0\rangle$

$-2\pi \le t \le 0$

$$\vec{F} = \langle 6\sin t + 0,\ -6\cos t + 0,\ 36\cos t\sin t \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{r}\,' \,dt$$

$$= \int_{-2\pi}^{0} \langle 6\sin t, -6\cos t, 36\cos t \sin t \rangle \cdot \langle -6\sin t, 6\cos t, 0 \rangle \,dt$$

$$= \int_{-2\pi}^{0} -36\sin^2 t - 36\cos^2 t \,dt = \int_{-2\pi}^{0} -36 \,dt = 72\pi$$