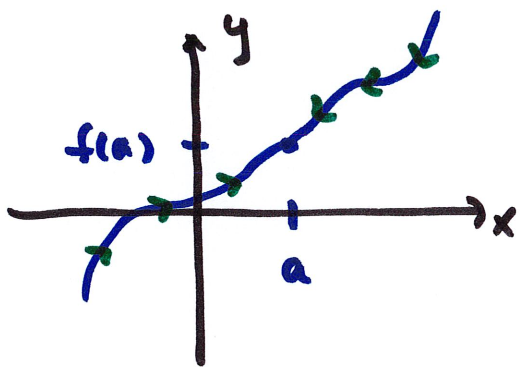
15.2 Limit and Continuity

recall if $\lim_{x \to a} f(x) = L$ means we can make $f(x)$ as closed to L as we want by making x sufficiently close to a

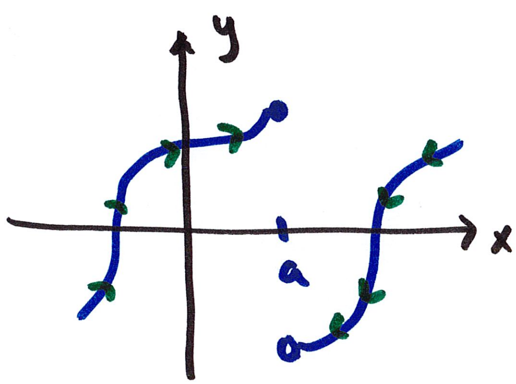
if the limit of $f(x)$ exists at $x = a$, then it doesn't matter whether we approach a from left or right

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = L$$



limit exists and is $f(a)$

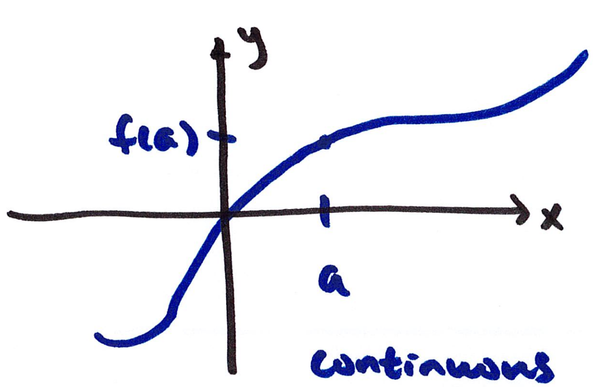
for the limit to exist at $x = a$, it should NOT matter how we approach.



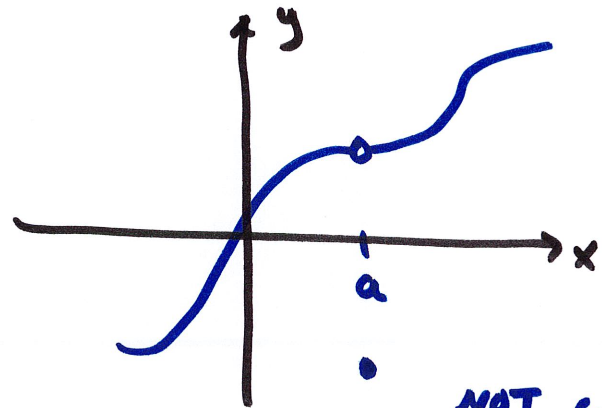
limit DNE

because $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

if $\lim_{x \to a} f(x) = f(a)$ then $f(x)$ is <u>continuous</u> at $x = a$



continuous at $x = a$



NOT continuous at $x = a$

we know many types of functions are <u>continuous</u>

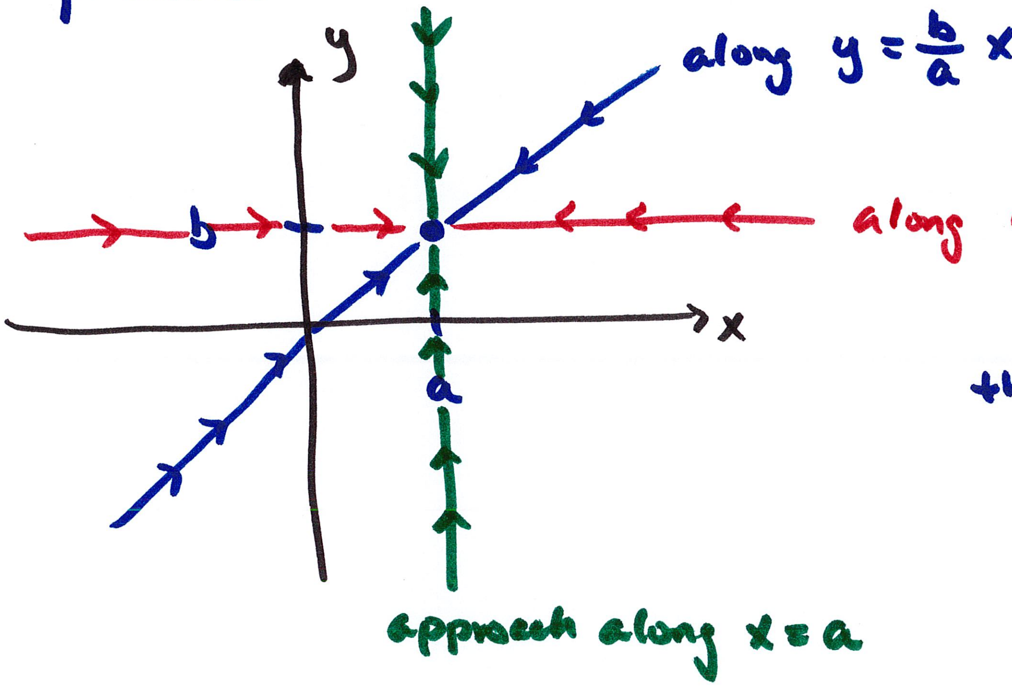
polynomial, sine, cosine, exponential $\rightarrow$ continuous everywhere

rational and logarithmic $\rightarrow$ continuous where defined

almost everything we know about limit and continuity carry over to functions of more variables

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ means we can make $f(x,y)$ as close to L as we want by making (x,y) close to (a,b)

for functions of more variables, <u>how</u> we approach (a,b) is more complicated.



there are MANY more possibilities!

eg. along a parabola

along an exponential

sine? cosine?

for the limit to exist at (a,b), it <u>should</u> <u>NOT</u> depend on the path taken.

example $\displaystyle\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$

before checking the possible paths, let's see if continuity can help

$\ln\left(\frac{1+y^2}{x^2+xy}\right)$ is logarithmic and is continuous wherever it is defined

$\downarrow$

$\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

is this function defined at $(1,0)$?

$\ln\left(\frac{1+0}{1+0}\right) = \ln(1) = 0$ so, yes.

therefore, because it is continuous, we know

$$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right) = f(1,0) = \boxed{0}$$

example $\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$

$\frac{y^2-4x^2}{2x^2+y^2}$ is rational so is continuous wherever defined

is $\frac{y^2-4x^2}{2x^2+y^2}$ defined at $(0,0)$? $\frac{0}{0}$ is not defined

so continuity does NOT help in finding the limit

now we must worry about how we got to $(0,0)$

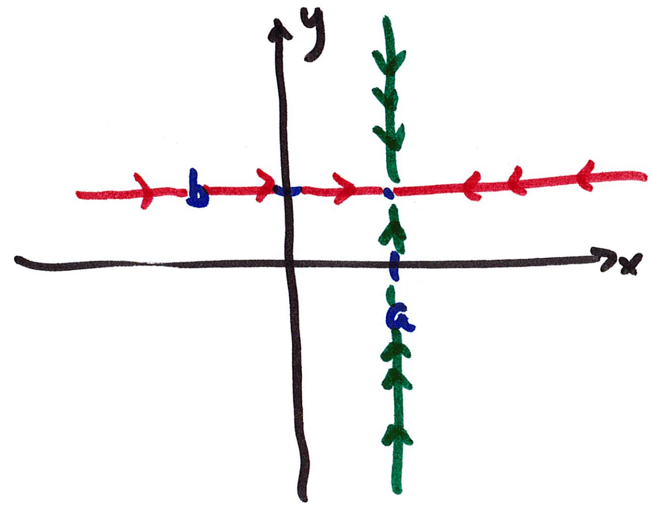
are there two paths that lead to different outcomes?

$\lim_{(x,y)\to(a,b)} \frac{y^2-4x^2}{2x^2+y^2}$

check the easy paths first

along $x=a$

along $y=b$

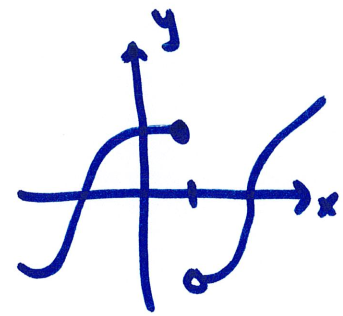


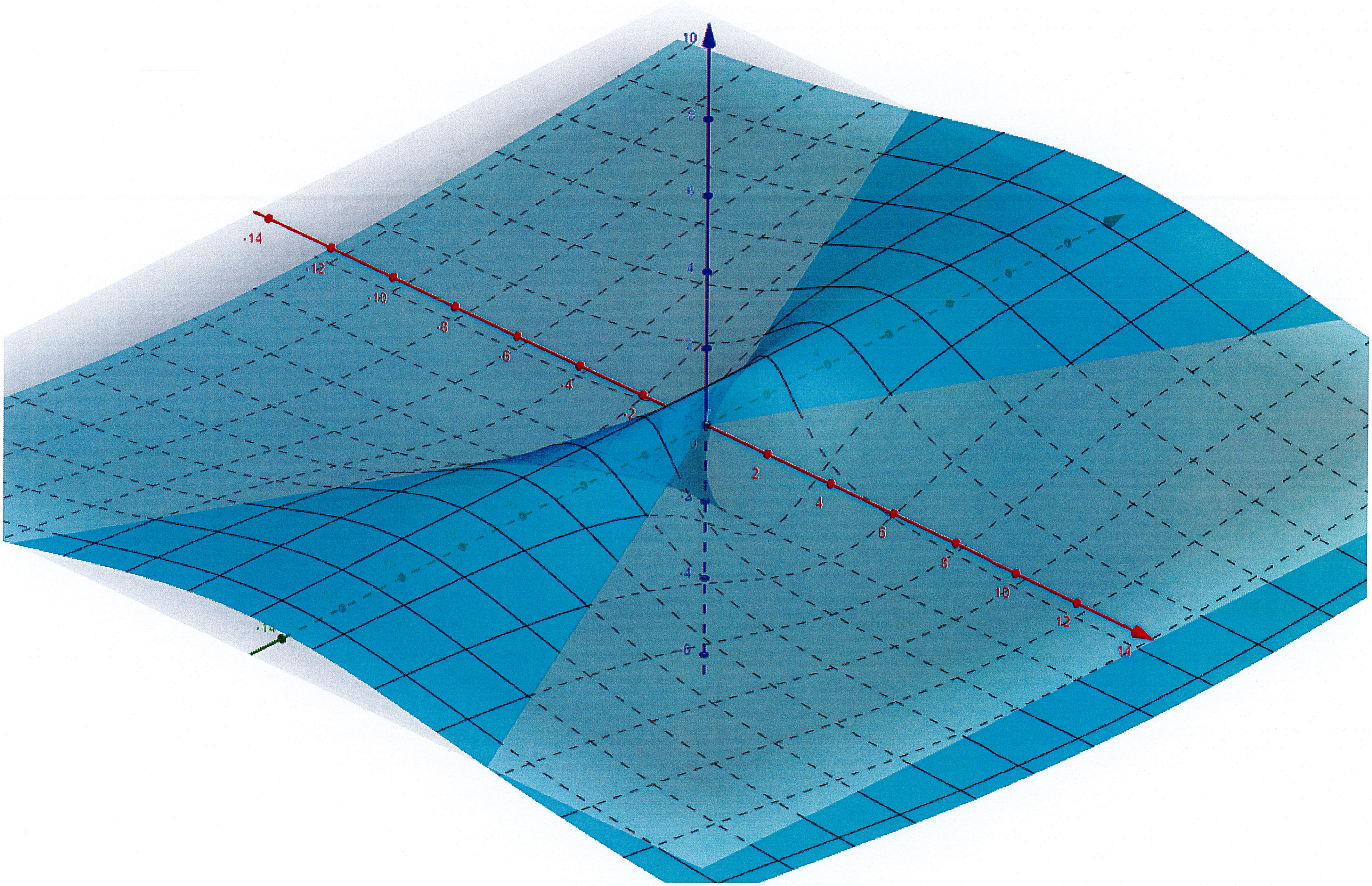
along $x=a=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-\overset{0}{4x^2}}{\underset{0}{2x^2}+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1$

along $y=b=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\overset{0}{y^2}-4x^2}{2x^2+\underset{0}{y^2}} = \lim_{x\to 0} \frac{-4x^2}{2x^2} = -2$

Since they don't match, the limit <u>DNE</u>

just like

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example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

clearly not defined at $(0,0)$

along $x=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{y\to 0} \frac{-y^2}{y} = \lim_{y\to 0} -y = 0$

along $y=0$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$

even though these match, this does NOT mean the limit exists (we only checked two out of ∞ paths)

what's next? try $y=x$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{x\to 0} \frac{x^2-x^2}{x+x} = 0$

Still NOT enough, only checked $3/\infty$

what's next? $y=x^2$, $y=x^3$, $y=e^x-1$, $y=\sin x$

NOT possible to check all

now try something that does NOT assume a path

back to $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x-y)}{\cancel{(x+y)}}$

$= \lim_{(x,y)\to(0,0)} (x-y) = 0$

0 0

this does NOT depend on HOW we approach (0,0)

so, limit is 0.

<u>example</u> $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+2y^2)}{x^2+2y^2}$ resembles $\lim_{x\to 0} \frac{\sin x}{x} = 1$

let $u = x^2+2y^2$

then $(x,y)\to(0,0)$ means $u\to 0$

$$\lim_{u\to 0} \frac{\sin(u)}{u} = 1$$

again, this makes no assumption of any particular path

so, limit is 1.