15.3 Partial Derivatives

recall if $y = f(x)$ then the derivative of $f(x)$ with respect to x

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

this also gives us the rate of change of y with respect to x

for a function of two variables, $z = f(x,y)$, z is affected by both x and y ~~and~~ they both contribute to the rate of change of z.

now we have more sources of change, we need to look at them individually first

$z = f(x,y)$

the partial derivative of z with respect to x is

script or funny ∂

$$\underbrace{\frac{\partial f}{\partial x}}_{\text{variable that is changing}} = f_{\underline{x}} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

y is held constant
only x is changing.

the partial derivative of z with respect to y

$$\frac{\partial f}{\partial y} = f_y = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

x is held constant
only y is changing

f_x : rate of change of f due to changing x only (y is held constant)

f_y : " " " " y " (x " " ")

in practice, we don't use the limit definition

we use the various rules we know

example $f(x,y) = x^2 + y^3 + xy$

keep y the same
treat y as a **constant**

the partial derivative of f with respect to x is

$$\frac{\partial}{\partial x}(x^2+y^3+xy)$$

y is constant

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^3) + \frac{\partial}{\partial x}(xy)$$

$$= 2x + 0 + y = \boxed{2x+y}$$

keep x the same
treat x as a **constant**

the partial derivative of f with respect to y

$$\frac{\partial}{\partial y}(x^2+y^3+xy)$$

const   const.

$$= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(xy)$$

$$= 0 + 3y^2 + x = \boxed{x+3y^2}$$

example $z = f(x,y) = x^3 + \tan(xy)$

y is const

$$\frac{\partial f}{\partial \underline{\underline{x}}} = \frac{\partial}{\partial x}\left(x^3 + \tan(xy)\right)$$

$$= 3x^2 + \frac{\partial}{\partial x}\tan(xy)$$

$$= \boxed{3x^2 + y\sec^2(xy)}$$

const.

$$\frac{\partial}{\partial x}\tan(xy)$$

$$= \sec^2(xy)\cdot\frac{\partial}{\partial x}(xy)$$

$$= \sec^2(xy)\cdot y$$

x is const

$$\frac{\partial f}{\partial \underline{\underline{y}}} = \frac{\partial}{\partial y}\left(x^3 + \tan(xy)\right)$$

$$= 0 + \sec^2(xy)\cdot\frac{\partial}{\partial y}(xy)$$

$$= \boxed{x\sec^2(xy)}$$

higher-order partial derivatives

example $f(x,y) = e^x \sin y$

y is const $\dfrac{\partial f}{\partial x} = f_x = e^x \sin y$

x is const $\dfrac{\partial f}{\partial y} = f_y = e^x \cos y$

} first-order partials

second-order partials

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}(e^x \cos y) = \cancel{e^x \cos y} - e^x \sin y$$

"mixed partials"

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underbrace{\frac{\partial^2 f}{\partial y\,\partial x}}_{\text{order is right to left}} = \underbrace{f_{xy}}_{\text{order is left to right}} = \frac{\partial}{\partial y}(e^x \sin y) = \underline{e^x \cos y}$$

} Same!

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = f_{yx} = \frac{\partial}{\partial x}(e^x \cos y) = \underline{e^x \cos y}$$

notice $f_{xy} = f_{yx}$, and that is not a coincidence

it turns out the "mixed partials" are always equal ~~iff~~ wherever $f(x,y)$ is defined and both f_x and f_y are continuous

example $\quad f(x,y) = e^{x^2 y}$

$$f_x = e^{x^2y} \cdot \frac{\partial}{\partial x}(x^2y) = 2xy\,e^{x^2y}$$

$$f_y = e^{x^2y} \cdot \frac{\partial}{\partial y}(x^2y) = x^2 e^{x^2y}$$

product rule

$$f_{xy} = \frac{\partial}{\partial y}\left((2xy)(e^{x^2y})\right) = (2xy)\frac{\partial}{\partial y}(e^{x^2y}) + (e^{x^2y})\frac{\partial}{\partial y}(2xy)$$

$$= (2xy)(x^2e^{x^2y}) + (e^{x^2y})(2x) = \boxed{2xe^{x^2y}(x^2y+1)}$$

$$f_{yx} = \frac{\partial}{\partial x}\left((x^2)(e^{x^2y})\right) = (x^2)\frac{\partial}{\partial x}(e^{x^2y}) + (e^{x^2y})\frac{\partial}{\partial x}(x^2)$$

$$= (x^2)(2xy\,e^{x^2y}) + (2x)(e^{x^2y}) = \boxed{2xe^{x^2y}(x^2y+1)}$$

example $f(x,y,z) = xyz$ we deal with it the same way as with $f(x,y)$

treat y, z as consts

$f_x = yz$

$f_y = xz$

$f_z = xy$

$f_{xy} = z$

$f_{yx} = z$

$f_{zx} = y$

$f_{xz} = y$

$f_{xyz} = \frac{\partial}{\partial z}(f_{xy}) = \frac{\partial}{\partial z}(z) = 1$

$f_{yxz} = \frac{\partial}{\partial z}(f_{yx}) = \frac{\partial}{\partial z}(z) = 1$

why?

$f_{xyz} = \frac{\partial}{\partial z}(\underline{\underline{f_{xy}}})$

$f_{yxz} = \frac{\partial}{\partial z}(\underline{\underline{f_{yx}}})$

$\rightarrow f_{xy} = f_{yx}$

third-order mixed partials are equal because the 2nd-order mixed are equal