Lesson 12 — 15.4 The Chain Rule

recall if $y = f(x)$ and $x = g(t)$ then to find the rate of change of y with respect to t, we used the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \text{or} \qquad y' = f'(g(t))\,g'(t)$$

$\frac{dy}{dx}$: how y is affected by x only

$\frac{dx}{dt}$: how x is changed by t only

look at how variables depend on one another in a <u>tree diagram</u>

$$\begin{tikzcd} y \arrow[d, dash] \arrow[d, bend left, "\frac{dy}{dx}"] \arrow[dd, bend right=60, "\frac{dy}{dt}"'] \\ x \arrow[d, dash] \arrow[d, bend left, "\frac{dx}{dt}"] \\ t \end{tikzcd}$$

each step down is a derivative

from top to bottom "chain" the steps together with more variables, there will be forks or branches in the tree

stepping down a branch → use partial derivative

example

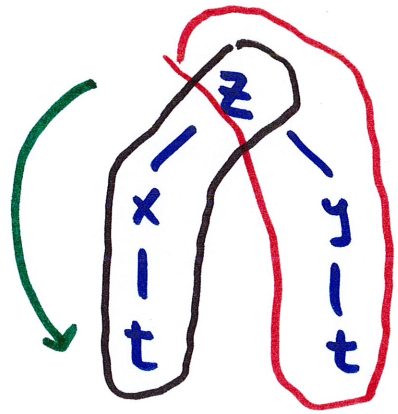
$$z = f(x, y) = x + y^2$$

and $x = e^t$, $y = \ln t$

find $\frac{dz}{dt}$

tree:

want! → $\frac{dz}{dt}$



Step down: two paths

black one: change in z w/ respect to t from x

red one: change in z w/ respect to t from y

$\frac{dz}{dt}$ = combine to the two paths

$$\frac{dz}{dt} = \underbrace{\frac{\partial z}{\partial x}\frac{dx}{dt}}_{\text{black branch}} + \underbrace{\frac{\partial z}{\partial y}\frac{dy}{dt}}_{\text{red branch}}$$

$z = x + y^2 \qquad x = e^t \qquad y = \ln t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\frac{dz}{d\boxed{t}} = (1)(e^t) + (2y)\left(\frac{1}{t}\right) = \boxed{e^t + \frac{2\ln t}{t}}$$

variable of interest
express answer in terms
of this

verify w/ answer from substituting x, y out first

$$z = x + y^2 = e^t + (\ln t)^2$$

$$\frac{dz}{dt} = e^t + 2(\ln t)\cdot\frac{1}{t} = e^t + \frac{2\ln t}{t} \qquad \text{same, of course}$$

example

$$z = \sin(x+y)$$
$$x = u^2 + v$$
$$y = 1 - 2uv$$

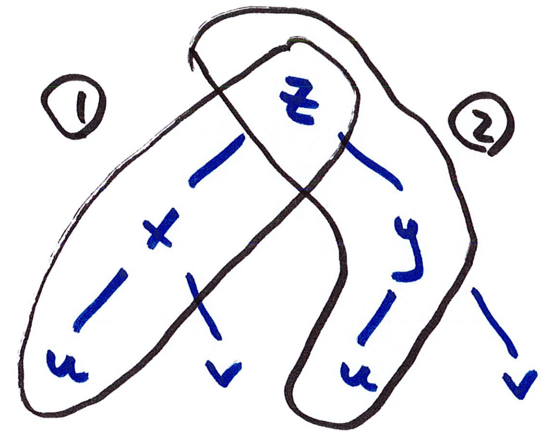
ultimately, z depends u and v

$$z = g(u,v)$$

find: $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$

note since z depends more than one thing ultimately, we use ∂ instead of d

tree:



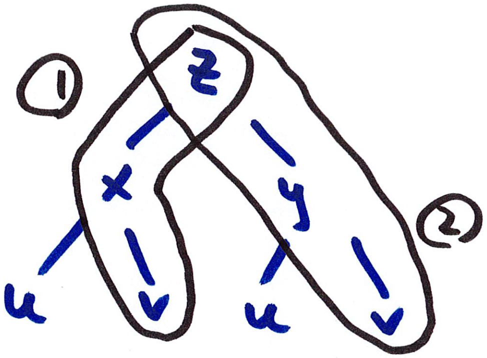
$$\frac{\partial z}{\partial u} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}_{2}$$

look for paths with this at bottom

$$= \cos(x+y)\cdot(2u) + \cos(x+y)\cdot(-2v)$$

write in terms of u

$$= \cos(u^2+v+1-2uv)(2u) + \cos(u^2+v+1-2uv)(-2v)$$
$$= \boxed{2(u-v)\cos(u^2+v+1-2uv)}$$



$$\frac{\partial z}{\partial v} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}_{2} = \cos(x+y)(1) + \cos(x+y)(-2u)$$

$$= \cos(x+y)(1-2u)$$

$$= \boxed{(1-2u)\cos(u^2+v+1-2uv)}$$

we can actually use this version of Chain Rule to perform implicit differentiation

$x^2+2y^2=4$ and y is an implicit function of x

find $\frac{dy}{dx}$

"old way": $\frac{d}{dx}(x^2+2y^2) = \frac{d}{dx}(4)$

$$2x + 4y \cdot \frac{dy}{dx} = 0$$

solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = -\frac{2x}{4y} = \boxed{-\frac{x}{2y}}$

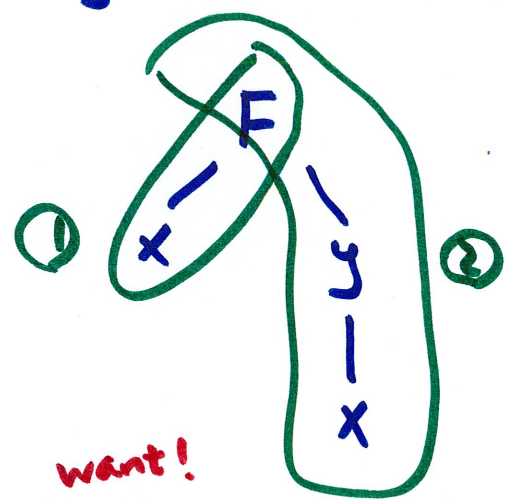
now let's try the same thing with the Chain Rule

$x^2+2y^2=4$ y is a function of x : $y=g(x)$

define a new function: $F(x,y) = x^2+2y^2-4 = 0 = f(x)$

note $F(x,y)$ is ultimately a function of x only because y is a function of x

now we see $f(x) = F(x,y) = 0$



we want to find $\frac{dy}{dx}$

want!

$$\frac{df}{dx} = \underbrace{\frac{\partial F}{\partial x}}_{1} + \underbrace{\frac{\partial F}{\partial y}\boxed{\frac{dy}{dx}}}_{2} = 0$$

because $f(x) = F(x,y) = 0$

solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \dfrac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \dfrac{-F_x}{F_y} = \dfrac{-2x}{4y} = \boxed{-\frac{x}{2y}}$ as expected

Same idea w/ more variables

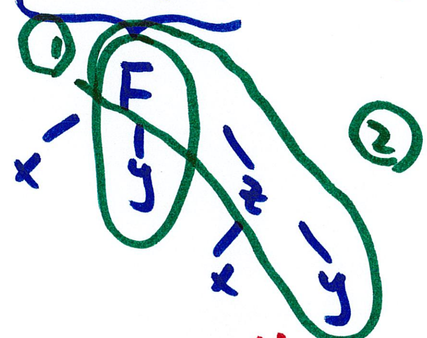
<u>example</u> $xy + yz + xz = 3$

z is some function of x, y $\rightarrow$ $z = g(x,y)$

find $\frac{\partial z}{\partial y}$

define $F(x,y,z) = xy + yz + xz - 3 = 0 = \underbrace{f(x,y)}_{\text{because } z \text{ is function of } x,\, y}$

$f(x,y) = \underbrace{F(x,y,z)} = 0$



want $\frac{\partial z}{\partial y}$ $\Rightarrow$ combine paths leading to y

$$\frac{\partial f}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\overset{\text{want!}}{\boxed{\frac{\partial z}{\partial y}}} = 0 \rightarrow \frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \boxed{\frac{-(x+z)}{y+x}}$$