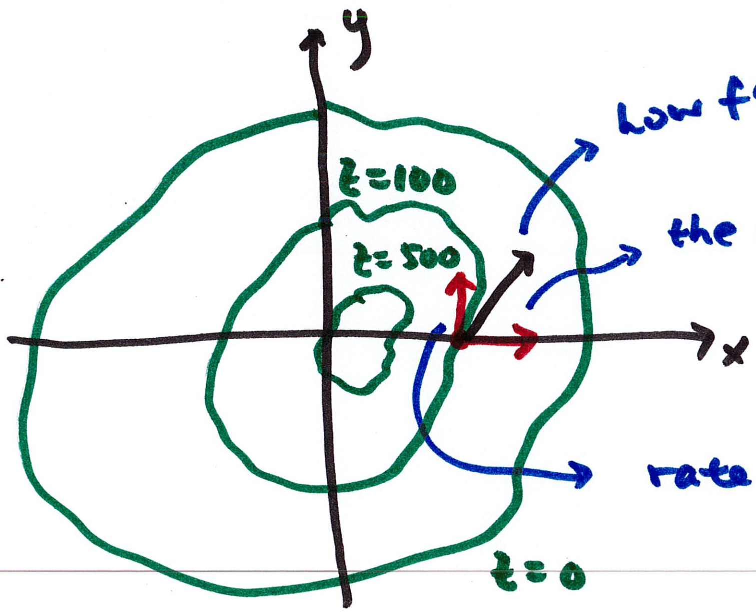
Lesson 13 15.5 Directional Derivative and the Gradient

$z = f(x,y)$

we know $\frac{\partial f}{\partial x} = f_x$ is the rate of change of $f(x,y)$ when only x changes

same idea with with $\frac{\partial f}{\partial y} = f_y$

an intuitive way to interpret: $z = f(x,y)$ is the surface of a hill / mountain



how fast is height changing?

the rate of change of height as we walk to the right (east)

is $\frac{\partial f}{\partial x}$

rate of change of height as we walk up (north)

is $\frac{\partial f}{\partial y}$

what about if we want the rate of change in a more general direction, for example, north-east?

the directional derivative can tell us that

let $\vec{u} = \langle a, b \rangle$ be a unit vector giving us the direction

then the rate of change of $z = f(x,y)$ is that direction is

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah,\, y+bh) - f(x,y)}{h}}$$

note if $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}$

$\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial y}$

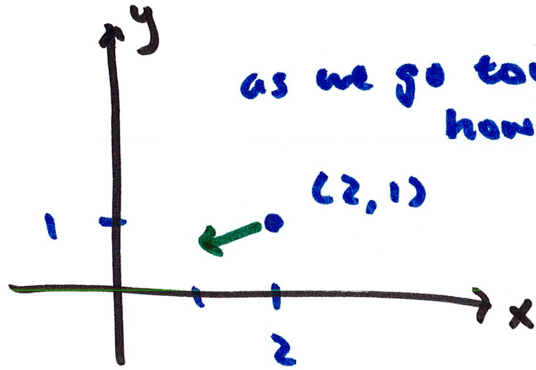
the more practical formula for the directional derivative is

$$\boxed{\begin{aligned} D_{\vec{u}} f(x,y) &= \frac{\partial f}{\partial x}(x,y)\, a + \frac{\partial f}{\partial y}(x,y)\, b \\ &= f_x a + f_y b \end{aligned}}$$

$\vec{u} = \langle a, b \rangle$ is a unit vector specifying the direction

example $f(x,y) = \cos(2x+3y)$

find the rate of change in the direction toward the origin starting at the point $(2,1)$



as we go toward origin how fast is $f(x,y) = \cos(2x+3y)$ changing?

first, find the unit vector specifying the direction from $(2,1)$ to $(0,0)$ $\rightarrow$ vector is $\langle -2, -1 \rangle$

but this is NOT a unit vector

so, $\vec{u} = \dfrac{\langle -2,-1 \rangle}{|\langle -2,-1\rangle|} = \langle \underbrace{\dfrac{-2}{\sqrt{5}}}_{a}, \underbrace{\dfrac{-1}{\sqrt{5}}}_{b} \rangle$

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$$= -2\sin(2x+3y)\cdot\frac{-2}{\sqrt{5}} - 3\sin(2x+3y)\cdot\frac{-1}{\sqrt{5}}$$

at $(2,1)$

$$= -2\sin(7)\cdot\frac{-2}{\sqrt{5}} - 3\sin(7)\cdot\frac{-1}{\sqrt{5}} = \boxed{\frac{7}{\sqrt{5}}\sin(7)}$$

so f is increasing at the rate of $\frac{7}{\sqrt{5}}\sin(7)$ as we head toward the origin starting at $(2,1)$

back to $D_{\vec{u}} f = f_x a + f_y b$

notice this is also $D_{\vec{u}} f = \underbrace{\langle f_x , f_y \rangle}_{\text{this vector of partial derivatives is called the Gradient}} \cdot \underbrace{\langle a, b \rangle}_{\vec{u} \text{ specifying the direction}}$

this vector of partial derivatives is called the **Gradient**

$\vec{u}$ specifying the direction

$$\boxed{\begin{aligned} &\text{gradient: } \vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \\ &\qquad\qquad\qquad\quad = \frac{\partial f}{\partial x}(x,y)\,\vec{i} + \frac{\partial f}{\partial y}(x,y)\,\vec{j} \end{aligned}}$$

"del"

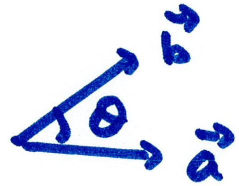
if $f(x,y,z) =$, then $\vec{\nabla} f = \langle f_x, f_y, f_z \rangle$

now we can rewrite the directional derivative as

$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

$\vec{\nabla} f$ is a vector: what do its direction and magnitude tell us?

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$ recall $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$



$$D_{\vec{u}} f = |\vec{\nabla} f| \underbrace{|\vec{u}|}_{1} \cos\theta$$

$$D_{\vec{u}} f = |\vec{\nabla} f| \underbrace{\cos\theta}_{\text{between } -1 \text{ and } 1}$$

this means the <u>maximum</u> ~~deriv~~ directional derivative

occurs when $\cos\theta = 1 \rightarrow \theta = 0 \rightarrow \vec{\nabla} f \parallel \vec{u}$

and the maximum value of $D_{\vec{u}} f$ is the magnitude of $\vec{\nabla} f$

so, the magnitude of gradient is the max $D_{\vec{u}} f$ and we go in the direction of $\vec{\nabla} f$ to see max $D_{\vec{u}} f$

steepest ascent

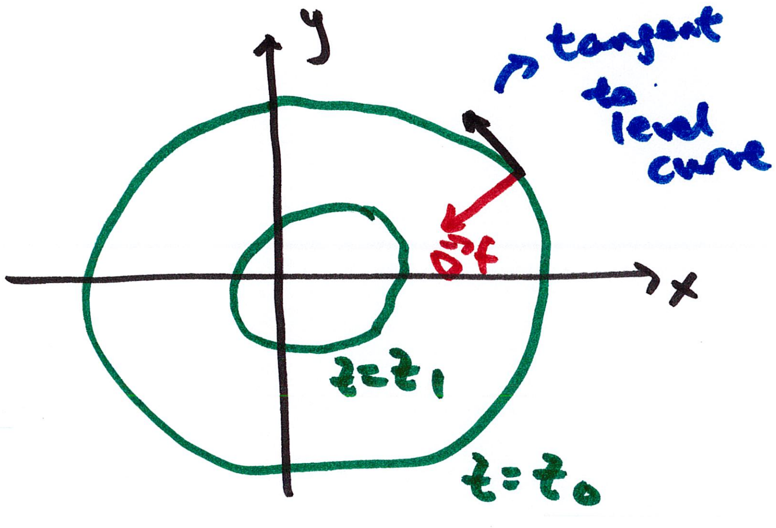
and to minimize $D_{\vec{u}} f$, we want $\cos\theta = -1 \rightarrow \theta = \pi$

so it's in the direction opposite to the gradient

steepest descent

note $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = 0$ if $\vec{\nabla} f \perp \vec{u}$

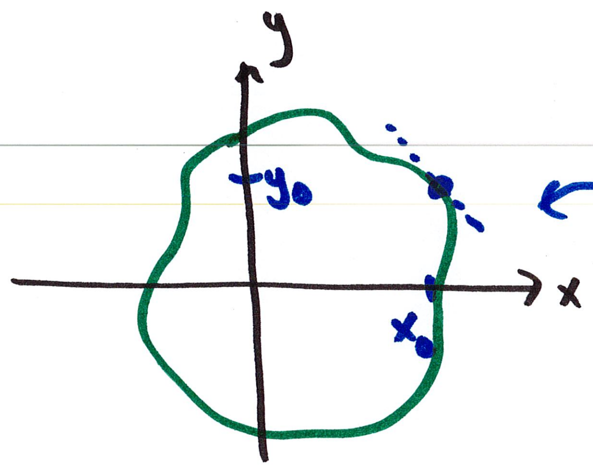
(go in the direction perpendicular to $\vec{\nabla} f$ to see no change in f)



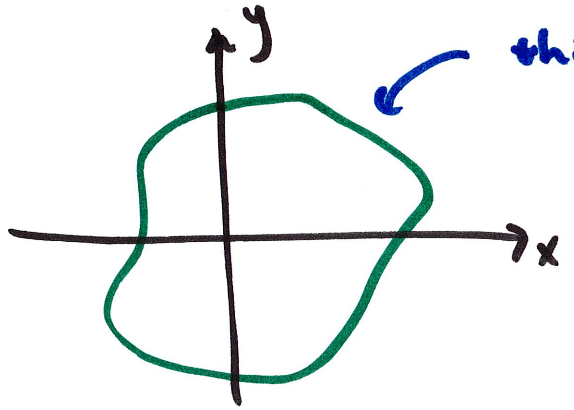
along a level curve, $f = z$ does NOT change.

so, $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = 0$ must mean we walk along a level curve (<u>tangent</u> to level curve)

$$\boxed{\text{therefore, since } \vec{\nabla} f \cdot \vec{u} = 0 \text{ means } \vec{\nabla} f \perp \vec{u}, \text{ the gradient MUST be perpendicular to a level curve}}$$



what is the slope at (x_0, y_0) on a level curve?



this level curve is $\vec{r}(t) = \langle x(t), y(t) \rangle$

then at (x_0, y_0),

$D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u} = 0$ along a level curve

so, $$D_{\vec{u}} f = 0 = \langle f_x, f_y \rangle \cdot \frac{\overbrace{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}^{\text{tangent vector along level curve}}}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|} = 0$$

$$\hookrightarrow \quad f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = 0$$

or $$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{f_x}{f_y}$$ by chain rule, left side is $\frac{dy}{dx}$

so, $$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$$ $\frac{dy}{dx}$ is slope of level curve

<u>example</u> $f(x,y) = xe^y$

find slope of level curve at $(1,2)$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{e^y}{xe^y} = -\frac{1}{x}$$

at $(1,2)$, $\frac{dy}{dx} = -1$

we also see that level curve is vertical whenever $x=0$ ($\frac{dy}{dx}$ is ~~undefind~~ undefined at $x=0$)