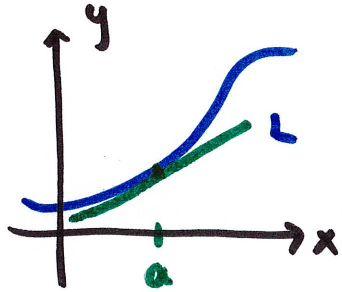
Lesson 14 15.6 The Tangent Plane and Linear Approximation

recall if $y = f(x)$

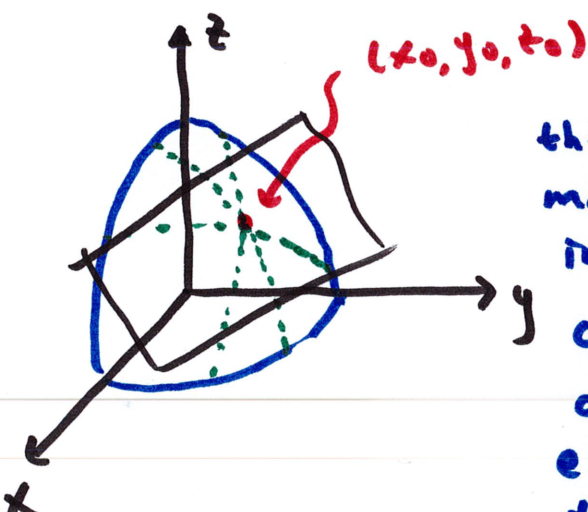


L: tangent line at $x = a$

$L = f(a) + f'(a)(x-a)$

near $x = a$, $L \approx f(x)$ (true curve)

$z = f(x,y)$ is a surface and at the point (x_0, y_0, z_0) there are infinitely-many tangent lines



the surface is made up of infinitely-many curves (green dash curves) each has a tangent line at (x_0, y_0, z_0)

all of these tangent lines form a tangent plane

It touches the surface at just one point (just like a tangent line) how to find equation of the tangent plane?

work with the idea that the surface is made up of curves

$\rightarrow \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

each has tangent vector $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

all of them live on the tangent plane

find a vector normal to ALL of them $\rightarrow$ normal vector of tangent plane

$z = f(x,y)$ is the surface

it contains a bunch of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

so, $z = f(x,y) \longrightarrow z(t) = f(x(t), y(t))$

let $F(x,y,z) = f(x(t), y(t)) - z(t) = 0$

$$\begin{tikzcd} & F \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & y \arrow[d, dash] & z \arrow[d, dash] \\ t & t & t \end{tikzcd}$$

by the Chain Rule,

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

because $F = f - z = \underline{\underline{0}}$

$$\underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\text{the gradient of } F \ (F = f - z)} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\text{tangent vectors on the tangent plane}} = 0$$

so, we see that $\vec{\nabla} F$ is normal to All tangent vectors (and the tangent plane)

so, $\vec{\nabla} F$ is the normal vector of the tangent plane

tangent plane at (x_0, y_0, z_0) is

$$\boxed{\begin{gathered} \langle F_x, F_y, F_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ \text{or } F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0 \\ \text{where } F = f - z \end{gathered}}$$

if we prefer to use $z = f(x,y)$ explicitly, then we tweak the equation above: $F = f - z$ so $F_x = f_x$, $F_y = f_y$, $F_z = -1$

$$\boxed{\begin{array}{l} f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = 0 \\ \text{or} \quad z - z_0 = f_x(x-x_0) + f_y(y-y_0) \end{array}}$$

<u>example</u> $z = f(x,y) = \sqrt{x^2+y^2}$ cone

find the tangent plane at $(3,4,5)$

let $F = f - z = \sqrt{x^2+y^2} - z$

$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$

$\vec{\nabla} F(3,4,5) = \left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle$

tangent plane eq: $\left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle \cdot \langle x-3, y-4, z-5 \rangle = 0$

$$\boxed{\frac{3}{5}(x-3) + \frac{4}{5}(y-4) - (z-5) = 0}$$

near $(3,4,5)$, the tangent plane $\approx$ true surface

the tangent plane eq: $z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) \approx \sqrt{x^2+y^2}$

how good is the approximation at $x = 3.01$, $y = 3.99$?

from tangent plane: $z = 5 + \frac{3}{5}(3.01-3) + \frac{4}{5}(3.99-4) = 4.998$

<u>true</u> value from $z = \sqrt{x^2+y^2} = \sqrt{(3.01)^2+(3.99)^2} = 4.99802$

the approximation is pretty good.

the approximation should be good if we stay close to the point of tangency $(3,4,5)$ in this example

no reason to expect the approximation to be even close if we go too far, say $x = 7$, $y = 9$

revisit the tangent plane eq: $z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

close to (x_0, y_0, z_0), $x - x_0 = dx$ (a very small change in x)

$$y - y_0 = dy$$

$$z - z_0 = dz$$

the TL eq. become: $\underbrace{dz = f_x\,dx + f_y\,dy}$

this allows us to estimate the change in z given changes in x and y

<u>example</u> $z = f(x,y) = x^2 y$

if x starts at 1 and increases by 0.01

if y " " 3 " decreases " 0.09

what is the approx. change in z?

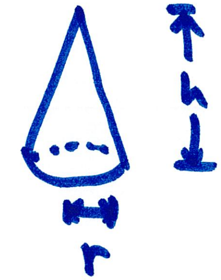
$$dz = f_x\,dx + f_y\,dy = (\boxed{2xy})\underset{\substack{=\\0.01}}{dx} + (\boxed{x^2})\underset{\substack{=\\-0.09}}{dy}$$

these x, and y refer to the starting values (where the tangent plane is formed)

$$= (2\cdot 1\cdot 3)(0.01) + (1^2)(-0.09)$$

$$= -0.03$$

example The volume of a cone is $V = \frac{1}{3}\pi r^2 h$



if r is increased by 1%

and h is decreased by 3%

what is the approx. % change in volume V ?

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh$$

$(dz = f_x\, dx + f_y\, dy)$

$$\underline{\underline{dV}} = \left(\frac{2}{3}\pi r h\right) \underline{\underline{dr}} + \left(\frac{1}{3}\pi r^2\right) \underline{\underline{dh}}$$

refer to absolute change, NOT relative change

DO NOT put in 1% for dr and −3% for dh

divide the eq. by $V = \frac{1}{3}\pi r^2 h$

relative change in V

$$\frac{dV}{V} = \frac{\frac{2}{3}\pi r h}{\frac{1}{3}\pi r^2 h} dr + \frac{\frac{1}{3}\pi r^2}{\frac{1}{3}\pi r^2 h} dh = 2\frac{dr}{r} + \frac{dh}{h}$$

relative (%) changes

now we get

$$\frac{dV}{V} = 2(\underbrace{0.01}_{1\%\text{ increase}}) + (\underbrace{-0.03}_{3\%\text{ decrease}}) = -0.01 = 1\% \text{ decrease}$$