Lesson 15 15.7 Maximum and Minimum Problems (part 1)

recall if $y=f(x)$, $f'(x)=0 \to$ critical numbers $x=c$
critical points $(c, f(c))$
possible locations of max/min of $f(x)$

the <u>Second Derivative Test</u>: if $f''(c)>0 \to$ relative <u>min</u> at $x=c$
concave up $\cup$
if $f''(c)<0 \to$ relative <u>max</u> at $x=c$
concave down $\cap$
if $f''(c)=0 \to$ test is inconclusive

we treat the function as if it were a parabola near a critical number
the basic idea for $z=f(x,y)$ is very similar.

$z = f(x,y)$

critical point: find (a,b) where BOTH $f_x(a,b)$ and $f_y(a,b)$ are zero

then, we form the discriminant: $D = f_{xx} f_{yy} - (f_{xy})^2$

if at the critical point (a,b)

$f_{xx} > 0$ and $D > 0$ then there is relative minimum at (a,b)

$f_{xx} < 0$ and $D > 0$ then there is relative maximum at (a,b)

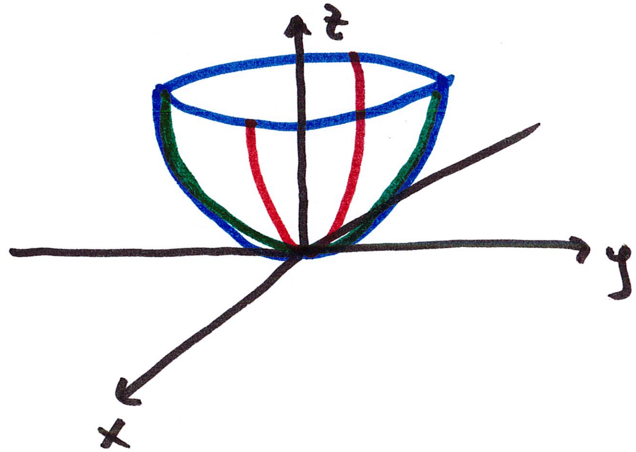
$D < 0$ then there is a saddle point at (a,b)

↳ (neither a max nor a min)

$D = 0$ then the test is inconclusive

↳ the point (a,b) could still be a max/min location this D tells us the shape of $z=f(x,y)$ near a critical point

for example, $f(x,y) = x^2+y^2$



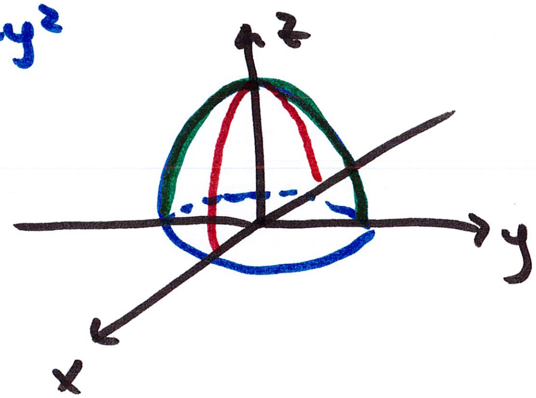
green: $f_{yy} > 0$ concave up

red: $f_{xx} > 0$ concave up

$D > 0 \to$ relative min at $(0,0)$

↳ and $f_{xx} > 0$ means locally the surface resembles an upward-opening paraboloid

$f(x,y) = 4 - x^2 - y^2$



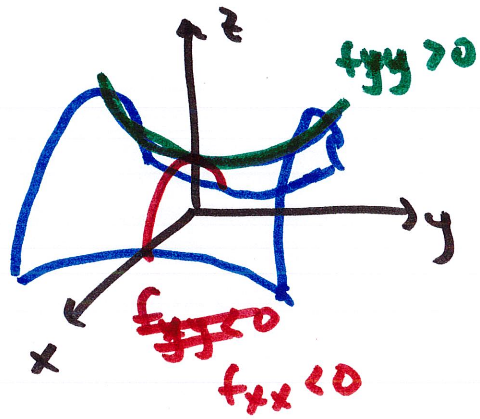
again, $\boxed{D > 0 \text{ and } f_{xx} < 0}$

max at $(0,0)$

↓

surface resembles a paraboloid opening down

$f(x,y) = -x^2+y^2$



$D < 0$

↳ one direction is upward the other is downward

<u>example</u> $f(x,y) = x^3 - 48xy + 64y^3$

find critical points: $f_x = 0$ and $f_y = 0$

$f_x = 3x^2 - 48y = 0$ — ①

$f_y = -48x + 192y^2 = 0$ — ②

from ①, $x^2 = 16y$

from ②, $\underline{x = 4y^2}$

$(4y^2)^2 = 16y$

$16y^4 = 16y$

$16y^4 - 16y = 0$

$16y(y^3 - 1) = 0$

$\underbrace{y = 0, \quad y = 1}$

these are the y-coordinates of critical pts

find their x-coords

$\boxed{\text{critical pts: } (0, 0), (4, 1)}$

now form $D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 6x, \quad f_{yy} = 384y \quad f_{xy}$ or $f_{yx} = -48$

$D = 2304xy - 2304$

at $(0,0)$, $D < 0 \rightarrow$ $\boxed{\text{saddle point at } (0,0)}$

at $(4,1)$, $D > 0$, $f_{xx} > 0 \rightarrow$ $\boxed{\text{relative min at } (4,1)}$

example $f(x,y) = xye^{-x^2-y^2}$

find critical pts: solve $f_x = 0$ AND $f_y = 0$

$$f_x = \cdots = ye^{-x^2-y^2}(-2x^2+1) = 0 \qquad ①$$

$$f_y = \cdots = xe^{-x^2-y^2}(-2y^2+1) = 0 \qquad ②$$

from ①: $\underbrace{(e^{-x^2-y^2})}_{\text{exponential which is NEVER zero}}\underbrace{(y)(-2x^2+1)}_{\text{one or both must be zero}} = 0$

so, $y = 0$ or $-2x^2+1 = 0 \to x = \pm\frac{1}{\sqrt{2}}$

do NOT pair these x and y to form critical pts because these came from ① alone they don't guarantee $f_y = 0$ (②)

from ①: $\boxed{y=0}$, $\boxed{x = \pm \frac{1}{\sqrt{2}}}$

from ②: $\underbrace{(e^{-x^2-y^2})}_{\neq 0}(x)(2y^2-1) = 0 \rightarrow \boxed{x=0}$, $2y^2-1=0$

$\boxed{y = \pm \frac{1}{\sqrt{2}}}$

pick one from each to form critical pts

$(0,0)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

then do the 2nd derivative test with $D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = \cdots = 2xy(2x^2-3)e^{-x^2-y^2}$

$f_{yy} = \cdots = 2xy(2y^2-3)e^{-x^2-y^2}$

$f_{xy} = \cdots = (2x^2-1)(2y^2-1)e^{-x^2-y^2}$

at $(0,0)$ $D < 0 \rightarrow$ saddle pt

at $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \rightarrow$ rel. max

at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \rightarrow$ rel. min what if $D = 0$? (test inconclusive)

example $f(x,y) = xy^2$

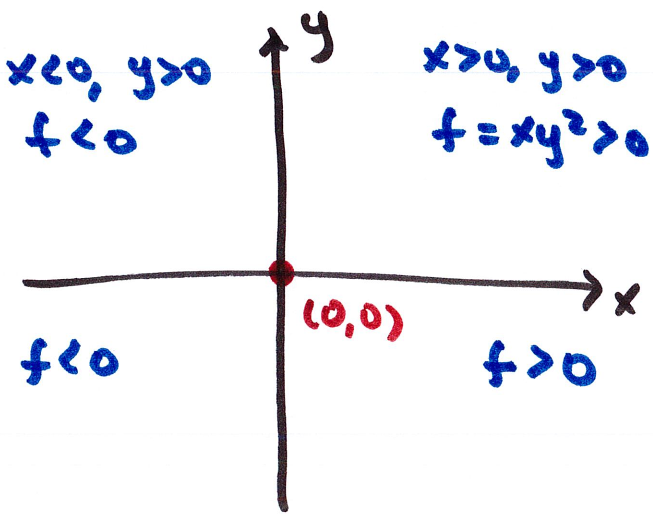
(0, 0) is the only critical pt

$f_x = y^2 \quad f_y = 2xy \quad f_{xy} = 2y \quad f_{yy} = 2x \quad f_{xx} = 0$

$D = -4y^2 = 0$ at $\underbrace{(0,0)}$

could still be max or min

how to classify?



if (0, 0) were a rel min, then ALL points nearly by should be higher than $f(0,0) = 0$

similarly for if (0,0) were a min

but the left shows (0,0) is neither max nor min