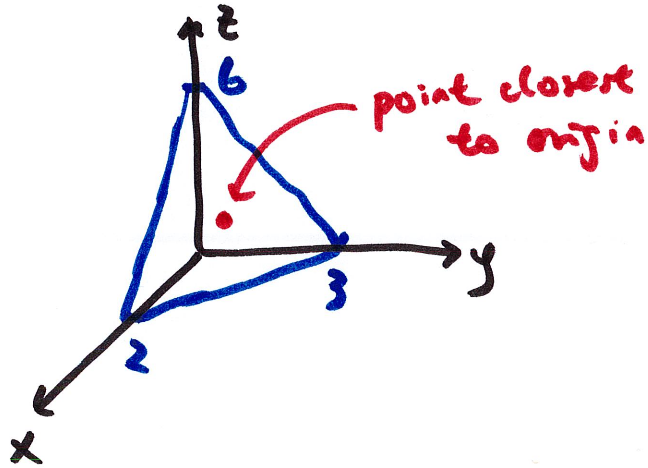
Lesson 16 15.7 Max/Min Problems (part 2)

example What point on the plane $3x+2y+z=6$ is the closest to the origin?



all points on plane (x, y, z)

and x, y, z must satisfy the plane eq: $3x+2y+z=6$

$(x, y, 6-3x-2y)$

distance to the origin $(0,0,0)$

$$d=\sqrt{(x-0)^2+(y-0)^2+(6-3x-2y-0)^2}$$

$$d=\sqrt{x^2+y^2+(6-3x-2y)^2}$$

minimize using the 2nd Derivative Test

note the ~~squa~~ square root makes the derivatives messy

since $d \geq 0$, we can minimize the square of it instead let $f = d^2 = x^2 + y^2 + (6 - 3x - 2y)^2$ now f_x, f_y, f_{xx}, etc are easier to find

find critical points: $f_x = 0$, $f_y = 0$

$f_x = 2x + 2(6-3x-2y)(-3) = 0 \longrightarrow 5x + 3y = 9$

$f_y = 2y + 2(6-3x-2y)(-2) = 0 \longrightarrow 6x + 5y = 12$

solving these simultaneously, we get

$x = \frac{9}{7}$, $y = \frac{6}{7}$ as our critical pt.

$$\left.\begin{array}{l} f_{xx} = 20 \\ f_{yy} = 10 \\ f_{xy} = 12 \end{array}\right\}$$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 200 - 144 > 0$

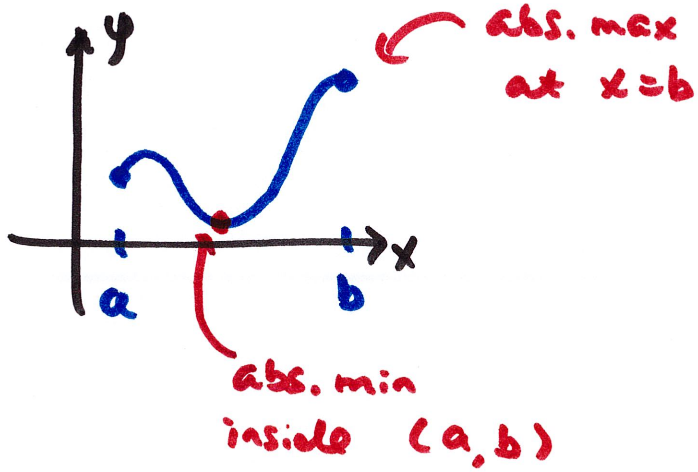
$D > 0$, $f_{xx} > 0$ at critical pt $x = 9/7$, $y = 6/7$

so that is where the minimum distance is

point: $x = 9/7$, $y = 6/7$, $z = 6 - 3x - 2y = 3/7$

point closest to origin is $\boxed{(9/7, 6/7, 3/7)}$

recall if $y=f(x)$, $a \le x \le b$ then the absolute max/min of $f(x)$ can be at a critical point inside $a \le x \le b$ or at the end points



procedure: find all interior critical pts

compare $f(x)$ at ↗

and at end pts and

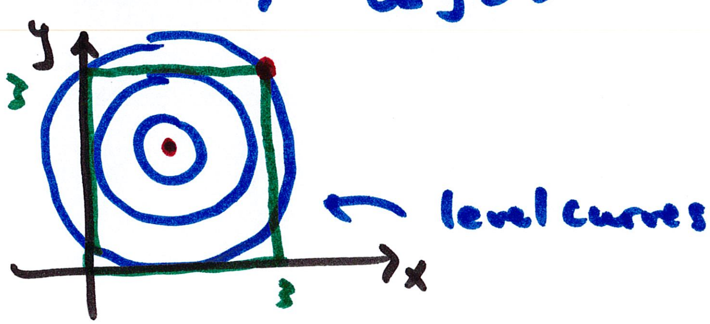
find the max/min $f(x)$

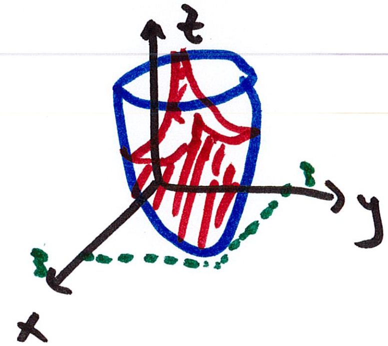
$z=f(x,y)$ is a surface, the domain restriction for (x,y) is a region on xy-plane, the boundaries of the region are curves

but the basic idea is the same: max/min of $z=f(x,y)$ at interior critical pts or boundary

$z=f(x,y)=(x-1)^2+(y-2)^2$

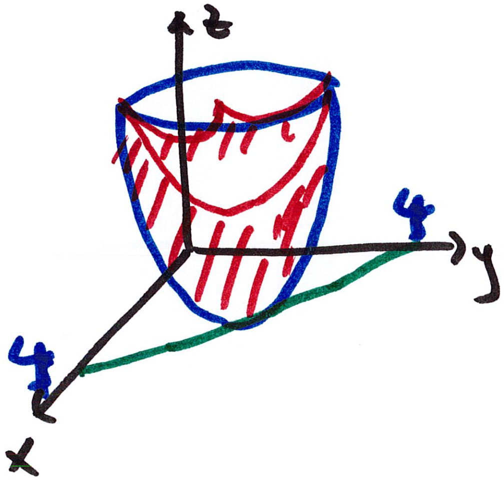
on $0 \le x \le 3$, $0 \le y \le 3$



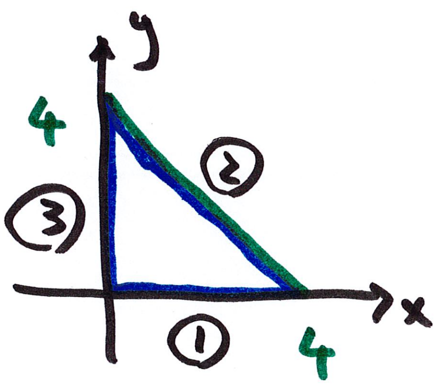


consider portion of paraboloid that casts a square shadow example $z = f(x,y) = x^2+y^2-2x-4y+10$

above the triangular region with vertices $(0,0)$, $(0,4)$, $(4,0)$



find the highest / ~~lowest~~ lowest points on the red portion (above the green triangle)



notice this can be broken down into 3 curves / lines

1. $y=0$, $0 \le x \le 4$
2. $y=4-x$, $0 \le x \le 4$
3. $x=0$, $0 \le y \le 4$

first, find critical pts within the triangular region

$$\left.\begin{array}{l} f_x = 2x-2 = 0 \\ f_y = 2y-4 = 0 \end{array}\right\} \quad \boxed{cp: (1,2)}$$

is this within the triangle w/ vertices (0,0), (0,4), (4,0) ?

yes, so we keep it

now we examine the boundaries to find where the max/min can be

1. $y=0, \quad 0 \le x \le 4$

$f(x,y) = x^2+y^2-2x-4y+10$ becomes $\underbrace{f(x) = x^2-2x+10, \quad 0 \le x \le 4}_{\text{this is now a one-variable max/min problem}}$

$f'(x) = 2x-2 = 0 \rightarrow x=1, y=0$

end pts: $x=0, \; y=0, \quad x=4, \; y=0$

$$\boxed{\begin{array}{l} \text{points of interest:} \\ (0,0), (4,0), (1,0) \end{array}}$$

on ②: $y = 4 - x, \quad 0 \le x \le 4$

$f(x,y) = x^2 + y^2 - 2x - 4y + 10$

becomes $f(x) = 2x^2 - 6x + 10, \quad 0 \le x \le 4$

$f'(x) = 4x - 6 = 0 \rightarrow x = 3/2, \quad y = 4 - x = 5/2$

end pts: $x = 0, \quad y = 4 - x = 4$

$x = 4, \quad y = 4 - x = 0$

$\boxed{\text{points of interest: } (0,4), (4,0), (3/2, 5/2)}$

on ③, following the same process, we get the following

$\boxed{\text{points of interest: } (0,2), (0,0), (0,4)}$

now we compare $f(x,y)$ at these points

points: $(1,2)$, $(0,0)$, $(4,0)$, $(1,0)$, $(0,4)$, $(3/2, 5/2)$, $(0,2)$

$f(1,2) = 5$ → abs. min at $x=1$, $y=2$, $z=5$ $(1,2,5)$

$f(0,0) = 10$

$f(4,0) = 18$ → abs. max at $x=4$, $y=0$, $z=18$ $(4,0,18)$

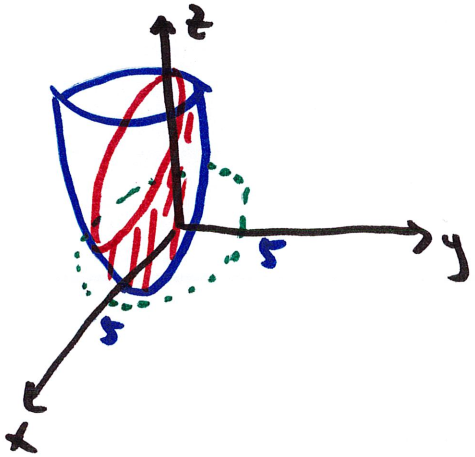
$f(0,4) = 10$

$f(3/2, 5/2) = 11/2$

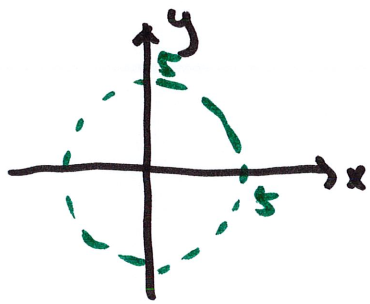
$f(0,2) = 6$

$f(1,0) = 9$

example Find abs. max/min of $f(x,y) = x^2+y^2-6x+9$ above the region $x^2+y^2 \le 25$



find the highest/lowest points on the part of the paraboloid directly above the green circle $(x^2+y^2=5)$



note we can't cut this into straight line segments like the previous example

but we can say

$y = \pm\sqrt{25-x^2}, \quad -5 \le x \le 5$

or $x = \pm\sqrt{25-y^2}, \quad -5 \le y \le 5$

back to $f(x,y) = x^2+y^2-6x+9$

find interior critical pts: $\left.\begin{array}{l} f_x = 2x-6 = 0 \rightarrow x=3 \\ f_y = 2y = 0 \rightarrow y=0 \end{array}\right\}$ $(3,0)$ inside circle? yes, so ~~keep~~; keep it now let's use $y = \pm\sqrt{25-x^2}$, $-5 \le x \le 5$ to describe the circle boundary

$f(x,y) = \underbrace{x^2+y^2}_{25} - 6x + 9$ becomes $f(x) = 25 - 6x + 9$, $-5 \le x \le 5$

because $x^2+y^2=25$ (circle)

$f'(x) = -6 \neq 0$ no critical pt

end pts: $x = -5$, $y = \pm\sqrt{25-x^2} = 0$

$x = 5$, $y = \pm\sqrt{25-x^2} = 0$

$\boxed{\text{points of interest: } (-5,0), (5,0)}$

compare $f(x,y) = x^2+y^2-6x+9$ at $\underbrace{(3,0)}_{\text{critical pt}}$, $(-5,0)$, $(5,0)$

$f(3,0) = 0$ abs. min

$f(-5,0) = 64$ abs. max

$f(5,0) = 4$