Lesson 17. 15.8 Lagrange Multipliers

(NOT on exam 1)

constrained optimization problems: find max/min of $f(x,y)$ subject to a certain condition

for example, find max/min of $f(x,y) = x^2+y^2$ subject to the condition that $xy = 1$

$f(x,y) = x^2+y^2$ is the objective (goal)

$g(x,y) = xy-1=0$ is called the constraint

if the functions are not two complicated, we can solve by substitution

$f(x,y) = x^2+y^2 \qquad xy=1 \rightarrow y=\frac{1}{x}$

$\hookrightarrow f(x,y) = x^2+\left(\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}$

$f'(x) = 2x-\frac{2}{x^3} = 0 \rightarrow x^4=1 \rightarrow x=1,-1$

since $y=\frac{1}{x}$, $y=1,-1$

critical pts: $(1,1)$, $(1,-1)$

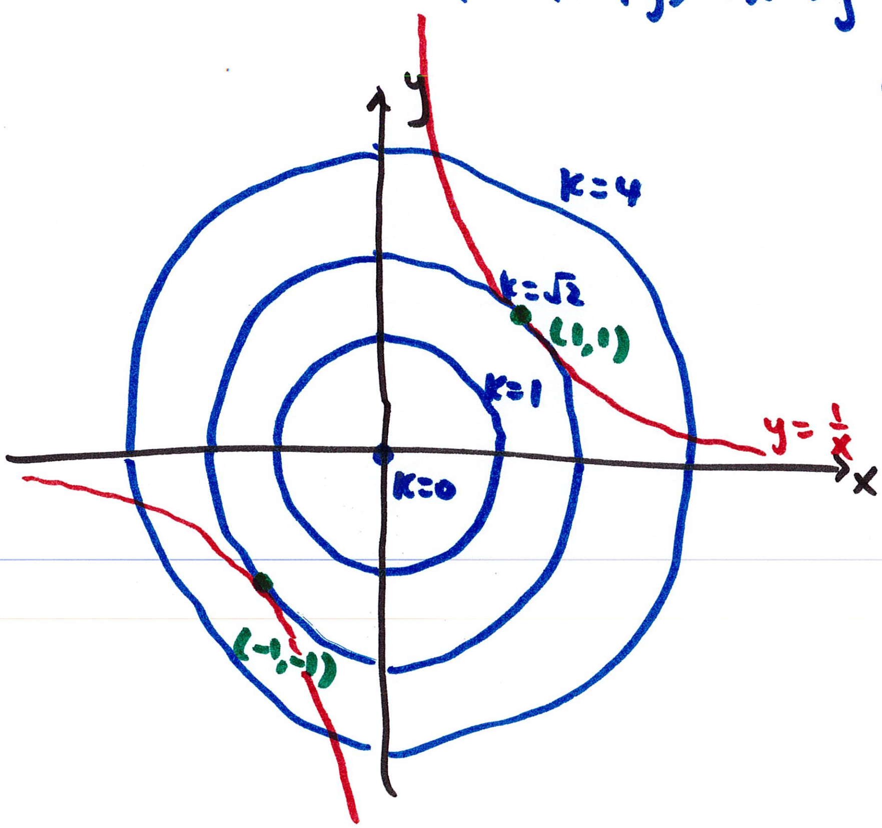
$f(x,y) = x^2+y^2$ $\quad xy=1$

does not have a max because x or y can be made arbitrarily big and therefore make $f(x,y)$ big

so $(1,1)$, $(-1,-1)$ must be locations of minimum of $f(x,y)=x^2+y^2$ subject to $xy=1$

the geometric interpretation is more important

level curves of $f(x,y)=x^2+y^2 \rightarrow x^2+y^2=k$



constraint : $xy=1 \rightarrow y=\frac{1}{x}$

(red hyperbolas)

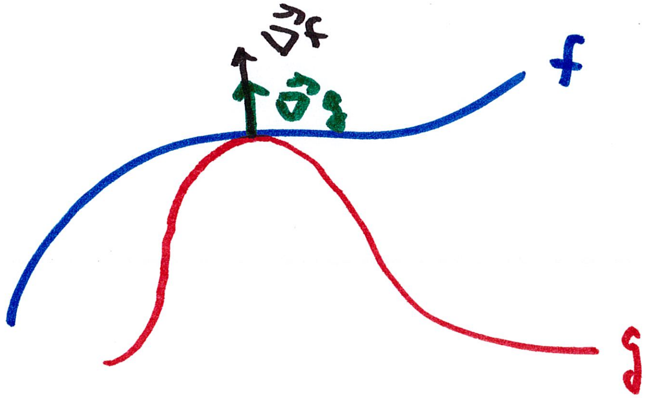
notice that at $(1,1)$ or $(-1,-1)$ any additional movement in any direction on the red curves result in higher k

but we want minimum k

notice at critical pts the level curve and constraint curve are <u>tangent</u> to each other at the point where the objective (f) and constraint (g) are tangent

$$\vec{\nabla} f \parallel \vec{\nabla} g \longrightarrow \vec{\nabla} f = \lambda \vec{\nabla} g$$

constant
Greek "lambda"



this means, if we want to find the constrained max/min of $f(x,y)$ subject to $g(x,y)$, ~~solve~~ we solve for locations where

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

this is called the Method of Lagrange Multipliers

<u>example</u> $f(x,y) = 4 - x^2 - y^2$

subject to the constraint $\underbrace{4x^2 + y^2 = 4}$

$g(x,y) = 4x^2 + y^2 - 4 = 0$

Solve: $\vec{\nabla} f = \lambda \vec{\nabla} g$

$\vec{\nabla} f = \langle -2x, -2y \rangle$

$\vec{\nabla} g = \langle 8x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow \langle -2x, -2y \rangle = \lambda \langle 8x, 2y \rangle$

$-2x = \lambda \cdot 8x$ — 1

$-2y = \lambda \cdot 2y$ — 2

1 $\lambda \cdot 8x + 2x = 0$

$2x(4\lambda + 1) = 0 \rightarrow 2x = 0$ or $\lambda = -1/4$

$x = 0$

if $x = 0$, we know y from $g(x,y)$

$0 + y^2 - 4 = 0 \rightarrow y = \pm 2$

$\boxed{\text{points of interest: } (0,2), (0,-2)}$

② $\lambda \cdot 2y + 2y = 0$

$2y(\lambda+1) = 0 \rightarrow 2y = 0, \quad \lambda = -1$

$\hookrightarrow y = 0$

from $g(x,y)$ we get $x = \pm 1$

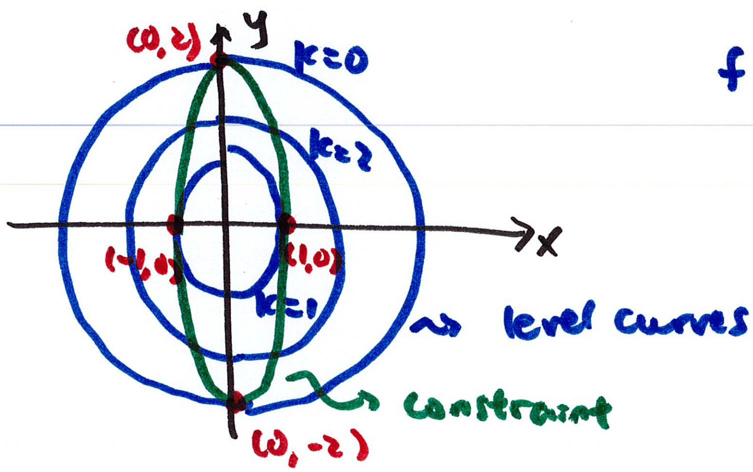
$\boxed{\text{points of interest: } (1,0), (-1,0)}$

now we compare $f(x,y)$ at these points

$f(x,y) = 4 - x^2 - y^2$

① $\left\{ \begin{array}{l} f(0,2) = 0 \\ f(0,-2) = 0 \end{array} \right\}$ 2 minima of $f = 0$ at $(0,2), (0,-2)$

② $\left\{ \begin{array}{l} f(1,0) = 3 \\ f(-1,0) = 3 \end{array} \right\}$ 2 maxima of $f = 3$ at $(1,0), (-1,0)$



$f = 4 - x^2 - y^2 \rightarrow x^2 + y^2 = (4-k)$

<u>example</u> $f(x,y,z) = xyz$

Subject to $x^2+y^2+z^2 = 3 \rightarrow g(x,y,z) = x^2+y^2+z^2-3=0$

geometric interpretation: find where on sphere of radius $\sqrt{3}$ with max/min of product of x, y, z coordinates

Solve $\vec{\nabla} f = \lambda \vec{\nabla} g$

$\vec{\nabla} f = \langle yz, xz, xy \rangle$

$\vec{\nabla} g = \langle 2x, 2y, 2z \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$

$$\left.\begin{array}{l} yz = \lambda \cdot 2x \;-\; (1) \rightarrow \lambda = \dfrac{yz}{2x} \\ xz = \lambda \cdot 2y \;-\; (2) \rightarrow \lambda = \dfrac{xz}{2y} \\ xy = \lambda \cdot 2z \;-\; (3) \rightarrow \lambda = \dfrac{xy}{2z} \end{array}\right\} \quad \frac{yz}{2x} = \frac{xz}{2y} = \frac{xy}{2z}$$

$$\underbrace{\frac{yz}{2x} = \frac{xz}{2y}} = \frac{xy}{2z}$$

$$\underbrace{y^2 z = x^2 z}_{x^2 = y^2} \qquad \underbrace{x z^2 = x y^2}_{y^2 = z^2}$$

$x^2 = y^2 = z^2 \to$ take into $g(x, y, z)$

$g(x,y,z) = x^2 + y^2 + z^2 - 3 = 0$

$x^2 + x^2 + x^2 = 3 \to x = \pm 1$

$y^2 = x^2 \to y = \pm 1$

$z^2 = y^2 \to z = \pm 1$

points of interest

$(1, 1, 1)$

$(-1, -1, -1)$

$(1, -1, -1)$, $(-1, -1, +1)$, $(-1, 1, -1)$

$(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$

plug into $f(x,y,z) = xyz$ to find

max: 1

min: -1