16.2 Double Integrals over General Regions

last time: over rectangular regions $R = \{(x,y) : a \le x \le b, \ c \le y \le d\}$

for rectangular regions, order is irrelevant

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

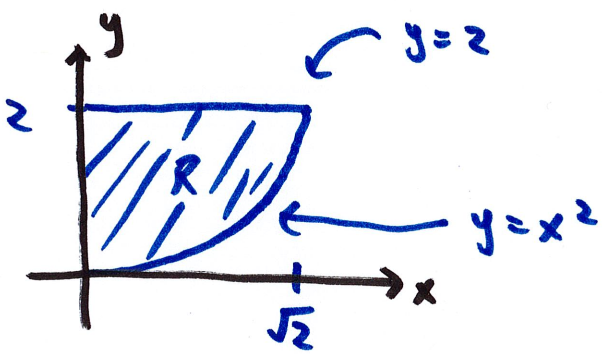
but swapping order <u>arbitrarily</u> is not possible for non-rectangular regions

<u>example</u> $f(x,y) = xy^2$ $\quad R = \{(x,y) : 0 \le x \le \sqrt{2}, \ x^2 \le y \le 2\}$

bottom border of R

top border

Step 1: sketch R



$$\boxed{\text{Basic rule: integrate the } \cancel{\text{constan}} \text{ variable with constant bounds } \underline{\underline{\text{LAST}}}}$$

So, here, we integrate x last (outside)

$$\int_0^{\sqrt{2}} \int_{x^2}^{2} xy^2 \, dy \, dx$$

$$x \qquad y$$

$$= \int_0^{\sqrt{2}} x \cdot \frac{y^3}{3} \Big|_{y=x^2}^{y=2} dx = \int_0^{\sqrt{2}} \left(\frac{8}{3}x - \frac{1}{3}x^7 \right) dx = \frac{4}{3}x^2 - \frac{1}{24}x^8 \Big|_0^{\sqrt{2}}$$

$$= \boxed{2}$$

what if we had integrated in the wrong order (variable w/ constant bounds not last)

$$\int_{x^2}^{2} \int_0^{\sqrt{2}} xy^2 \, dx \, dy = \int_{x^2}^{2} \frac{1}{2}x^2y^2 \Big|_{x=0}^{x=\sqrt{2}} dy$$

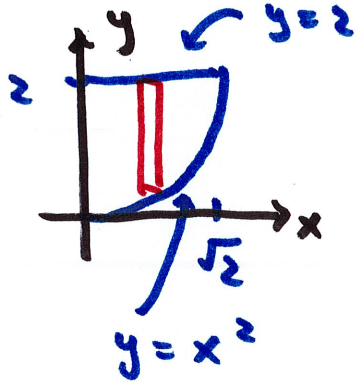
$$y \qquad x$$

$$= \int_{x^2}^{2} y^2 \, dy = \frac{1}{3}y^3 \Big|_{x^2}^{2} = \frac{8}{3} - \frac{x^6}{3}$$

what to do w/ this?

we can still swap the order, but we need to reformulate the region correctly

$R = \{(x,y) : \ 0 \le x \le \sqrt{2}, \ \ x^2 \le y \le 2\}$



this is sometimes called a <u>Type I</u> region

→ x bounded by constants, y by non constants

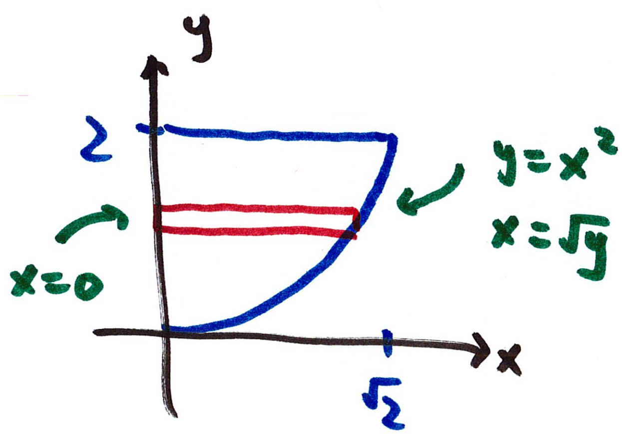
$x^2 \le y \le 2$ represents the height of a small rectangle that makes up the region

bottom of rectangle

top

to swap order correctly, we need to change to a <u>Type II</u> region

→ y bounded by constants
x by non constants



$0 \le y \le 2$

$0 \le x \le \sqrt{y}$

left

right border

$$\boxed{R = \{(x,y) : \ 0 \le x \le \sqrt{y}, \ \ 0 \le y \le 2\}}$$

integral w/ new formulation

$$\int_0^2 \int_0^{\sqrt{y}} xy^2 \, dx \, dy = \int_0^2 \frac{1}{2}x^2y^2 \Big|_{x=0}^{x=\sqrt{y}} dy$$

$y \qquad x$

$$= \int_0^2 \frac{1}{2} y^3 \, dy = \frac{1}{8} y^4 \Big|_0^2 = \boxed{2}$$

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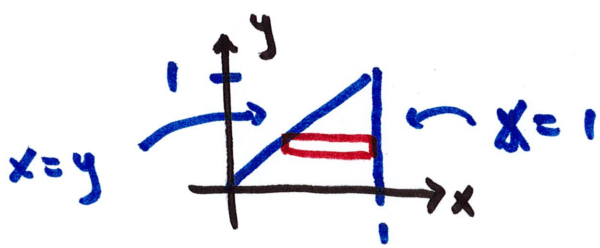
example $\int_0^1 \int_y^1 e^{x^2} dx \, dy$

left $\quad$ right

as given: $R = \{(x,y): \; y \le x \le 1, \quad 0 \le y \le 1\}$ Type II

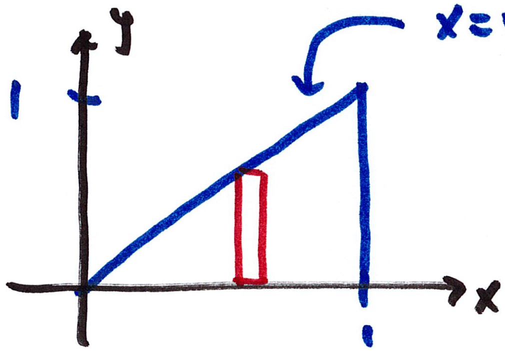


can we stick w/ this formulation?

$$\int_0^1 \int_y^1 e^{x^2} dx \, dy$$

first round: $\int_y^1 e^{x^2} dx$

this is not an integral we can do by hand we can't deal with R as a Type II region, let's swap to Type I



$x = y \longleftrightarrow y = x$

$$\overset{\text{bottom}}{0} \le y \le \overset{\text{top}}{x}$$

$$0 \le x \le 1$$

$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x\}$$

Integral: $$\underset{x}{\int_0^1} \underset{y}{\int_0^x} e^{x^2}\, dy\, dx = \int_0^1 y e^{x^2} \Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 x e^{x^2}\, dx = \cdots = \boxed{\frac{1}{2}(e-1)}$$

Subs: $u = x^2$

$du = 2x\, dx$

example

$$\underbrace{\int_{e^{-2}}^{1}\int_{-\ln y}^{2} f(x,y)\,dx\,dy}+\underbrace{\int_{1}^{e^{2}}\int_{\ln y}^{2} f(x,y)\,dx\,dy}$$

turn this into an integral with the other order

$e^{-2} \le y \le 1$

$-\ln y \le x \le 2$

left   right

left curve

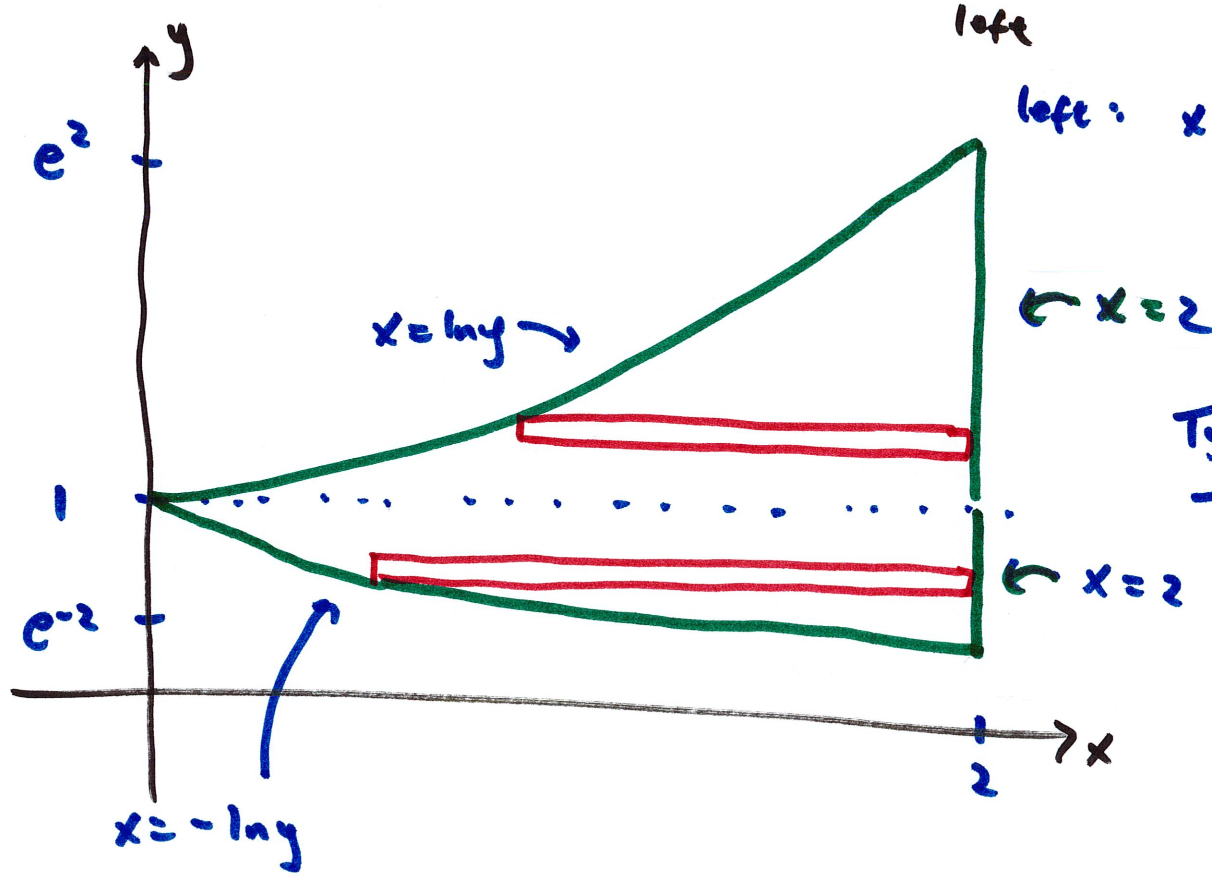
$x = -\ln y$

$1 \le y \le e^{2}$

$\ln y \le x \le 2$

left   right

left : $x = \ln y$

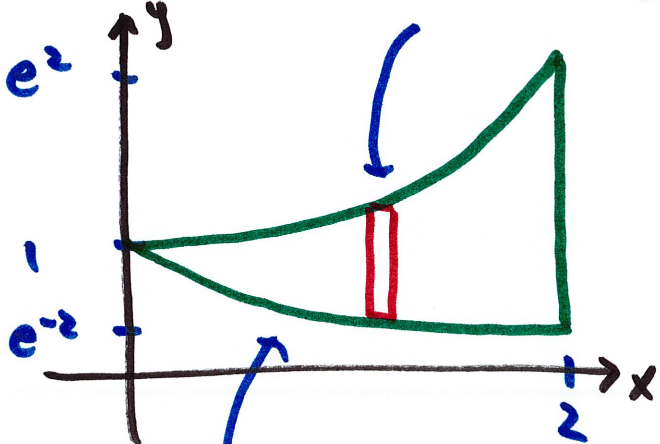


Type II

change to Type I :

$x = \ln y \Leftrightarrow y = e^x$

Since the top and bottom of rectangle always touch the same curve, only one integral is needed



$x = -\ln y$

$-x = \ln y$

$y = e^{-x}$

$$\underset{\text{curve}}{e^{-x}} \le y \le \underset{\text{top}}{e^x}$$

$$0 \le x \le 2$$

$$R = \{(x,y) : 0 \le x \le 2,\ e^{-x} \le y \le e^x\}$$

new integral : $\displaystyle\underbrace{\int_0^2}_{x} \underbrace{\int_{e^{-x}}^{e^x}}_{y} f(x,y)\, dy\, dx$