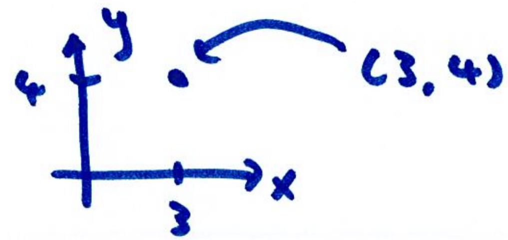
16.3 Double Integrals in Polar Coordinates

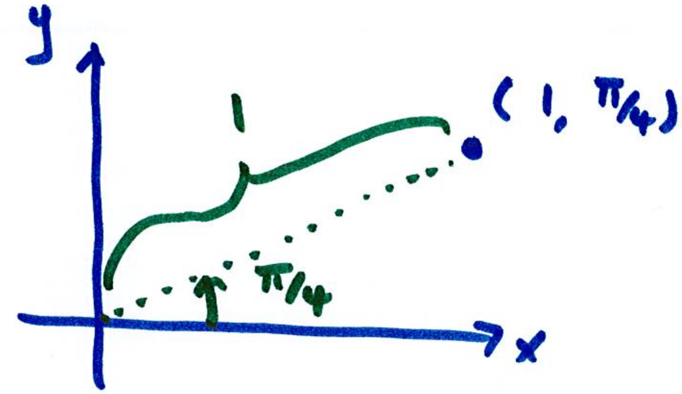
in rectangular or Cartesian coordinates, a point is (x, y)



polar : (r, θ)

r : displacement from origin

θ : angle of line from origin to the point formed with x-axis

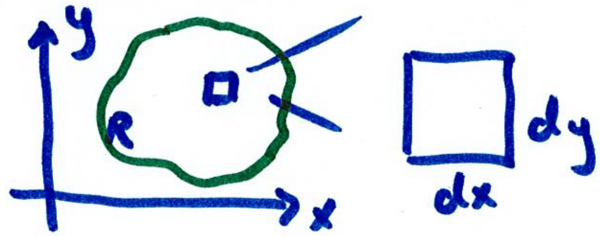


conversion : $x^2+y^2=r^2$

$x = r\cos\theta$

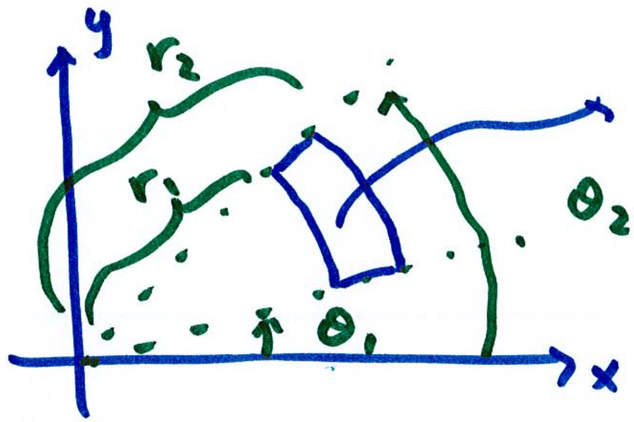
$y = r\sin\theta$

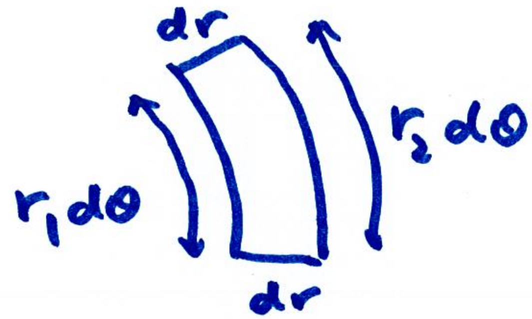
In Cartesian, $\iint\limits_R f(x,y)\,dA$ $\qquad dA = dx\,dy = dy\,dx$



dA : area of a small patch of R

in polar: $\iint_R f(r,\theta)\, dA$ what is dA in polar?





dr: difference between r_1 and r_2

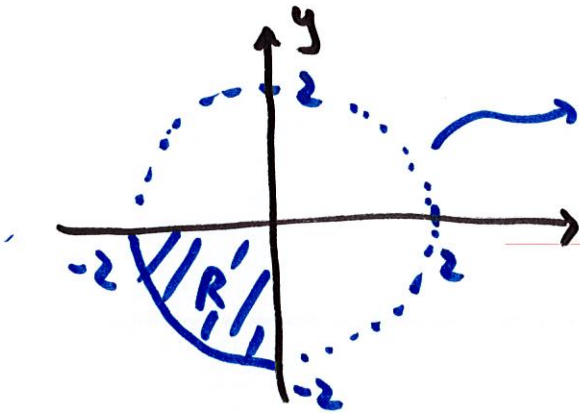
$d\theta$: difference between θ_1 and θ_2

if dr and $d\theta$ are very small, then $r_1 \approx r_2 = r$

so, dA in polar is

$$\boxed{dA = r\, dr\, d\theta}$$

example $\iint\limits_R \cos(x^2+y^2)\,dA$ R: circle centered at $(0,0)$ radius 2 in 3rd quadrant only



$x^2+y^2=4$

$y=\sqrt{4-x^2}$ upper half

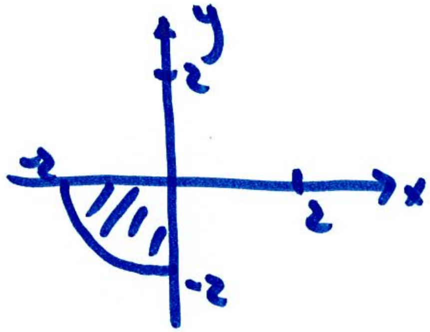
$y=-\sqrt{4-x^2}$ lower half

R is in QIII only, so as a Type I region

$$R=\{(x,y): -2\le x\le 0, -\sqrt{4-x^2}\le y\le 0\}$$

integral: $\displaystyle\int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{0}\cos(x^2+y^2)\,dy\,dx$ difficult in Cartesian

change to polar: good with w/ circles or circle-like R expressed in polar:



$$R = \{ (r,\theta) : \underbrace{0 \le r \le 2}_{\substack{\text{circle} \\ \text{radius } 2}}, \ \underbrace{\pi \le \theta \le \tfrac{3\pi}{2}}_{Q\text{III}} \}$$

$$\iint_R \cos(\underbrace{x^2+y^2}_{r^2})\, dA = \underset{\theta}{\int_{\pi}^{3\pi/2}} \underset{r}{\int_0^2} \cos(r^2)\, \underbrace{r\, dr\, d\theta}_{dA}$$

dA: $r\,dr\,d\theta$

$$= \int_{\pi}^{3\pi/2} \int_0^2 r\cos(r^2)\, dr\, d\theta$$

subs: $u = r^2$
$du = 2r\,dr$

$$= \cdots = \boxed{\frac{\pi}{4}\sin(4)}$$

<u>example</u> $\displaystyle\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$

terrible in Cartesian

the bounds of y strongly suggest R is circle or circle-like

polar is good for that

$$0 \le x \le 1$$

$$\underbrace{\sqrt{x-x^2}} \le y \le \underbrace{\sqrt{1-x^2}}$$

$y = \sqrt{x-x^2}$

$y^2 = x - x^2$

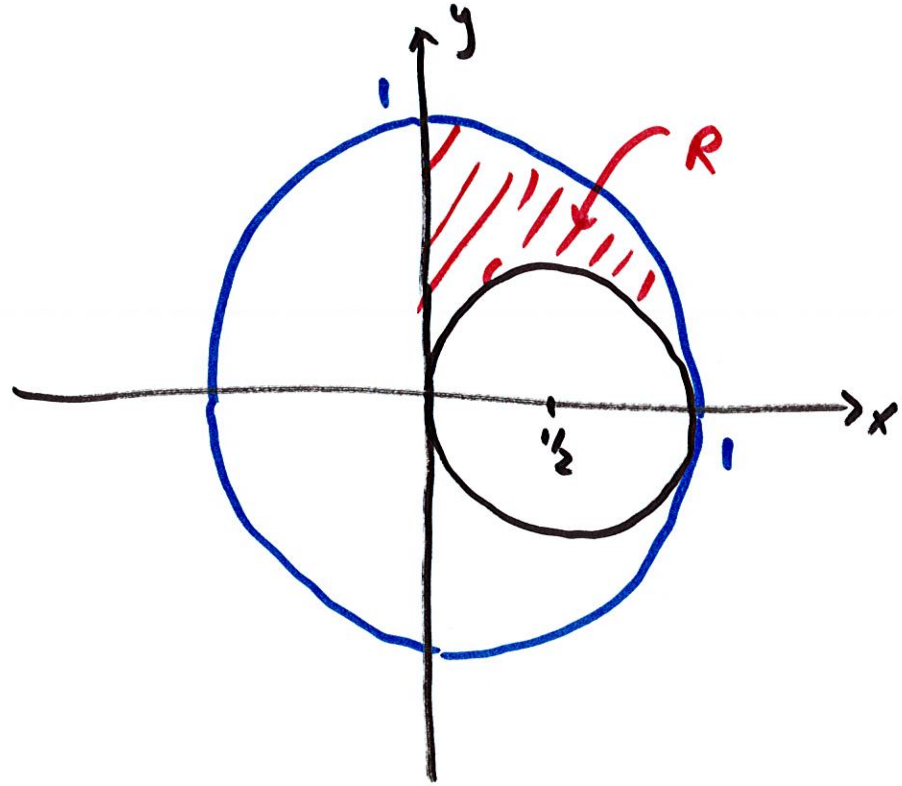
$x^2 - x + y^2 = 0$

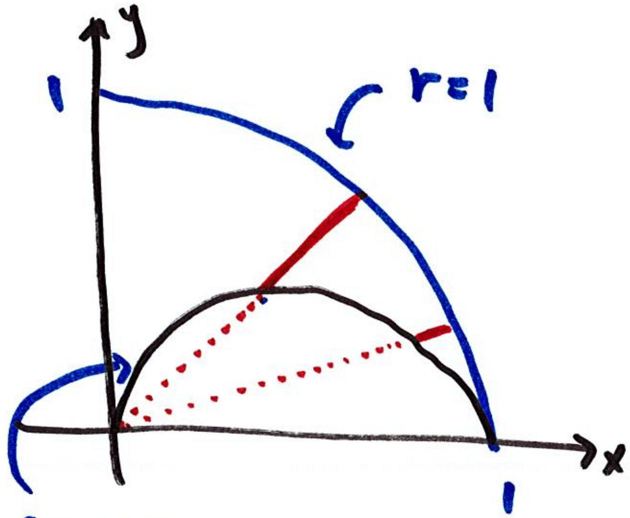
$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$

circle radius $1/2$

center : $(\frac{1}{2}, 0)$

upper half of circle radius 1 center: $(0,0)$





equation in polar ?

bounds for θ is easy

$$\boxed{0 \le \theta \le \pi/2}$$

bounds for r: draw a line from origin to edge of R then see portion inside R

inner circle: $y^2 = x - x^2$

$$\underbrace{x^2+y^2}_{r^2} = \underbrace{x}_{r\cos\theta}$$

$$r^2 = r\cos\theta$$

$$r = \cos\theta$$

we want r to go from inner circle to outer circle

so $\boxed{\cos\theta \le r \le 1}$

$$R = \{(r,\theta) : \underbrace{\cos\theta \le r \le 1}_{\text{at least one non constant}},\ \underbrace{0 \le \theta \le \pi/2}_{\text{constant bound}}\}$$

integrate θ last

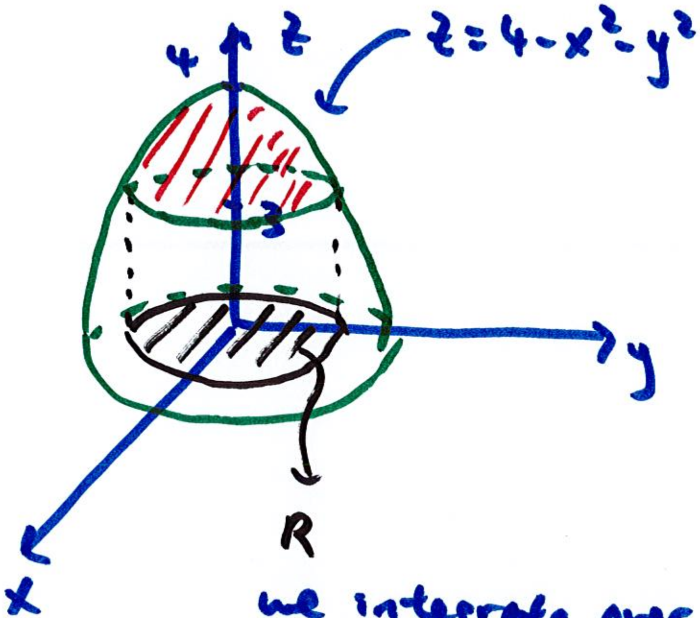
integral in polar:

$$\underset{\theta}{\int_{0}^{\pi/2}} \underset{r}{\int_{\cos\theta}^{1}} \underset{x^2+y^2}{(r^2)}^{3/2} \, \underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_{0}^{\pi/2} \int_{\cos\theta}^{1} r^4 \, dr\, d\theta = \cdots = \boxed{\frac{\pi}{10} - \frac{8}{75}}$$

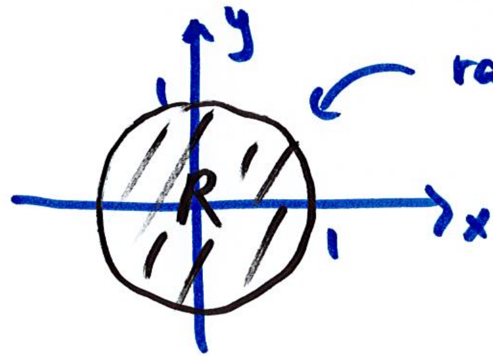
example Find the volume of the solid bounded above by

$z = 4 - x^2 - y^2$ ~~above~~ and below by $z = 3$



we want to integrate the height from $z = 3$ to $z = 4 - x^2 - y^2$ above the region that is the projection of the base of the solid we want

we integrate over the "shadow" of the top of dome



radius ?

$z = 4 - x^2 - y^2$
at $z = 3$
$3 = 4 - x^2 - y^2$
$x^2 + y^2 = 1$

so radius is 1

$$\boxed{R = \{(r, \theta) : 0 \le r \le 1, \quad 0 \le \theta \le 2\pi\}}$$

now we integrate $\cancel{z=}\ \underbrace{(4-x^2-y^2)}_{\text{paraboloid}} - \underset{\text{plane at } z=3}{\underset{\downarrow}{3}}$ over R

$$1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$$

Volume: $$\underset{\theta}{\int_0^{2\pi}} \underset{r}{\int_0^{1}} (1-r^2)\, \underbrace{r\,dr\,d\theta}_{dA} = \dots = \boxed{\frac{\pi}{2}}$$