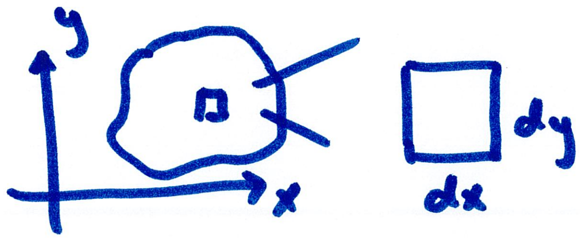
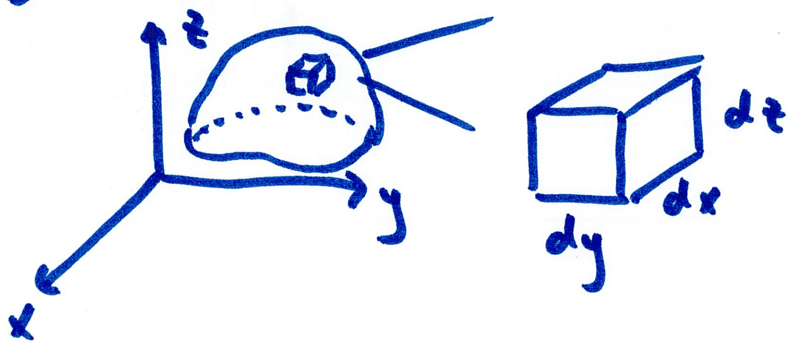
16.4 Triple Integrals

$\iint_R f(x,y)\,dA$ is the accumulation of $f(x,y)$ all over the region R.



$dA = dx\,dy = dy\,dx$ two possible orders

$\iiint_D f(x,y,z)\,dV$ accumulates $f(x,y,z)$ all over the volume D

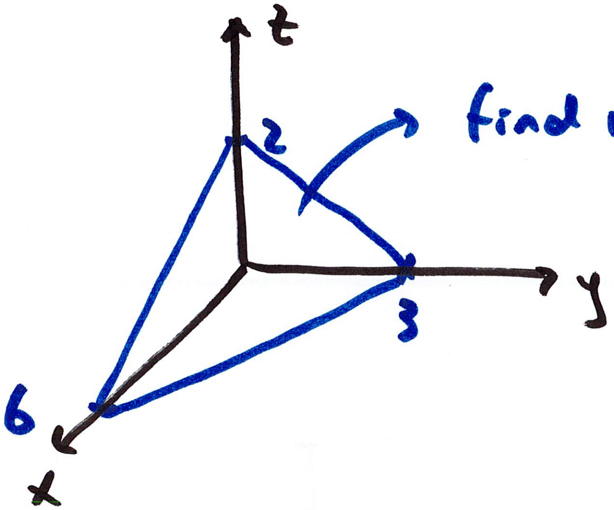


$$\left.\begin{aligned} \text{volume of } dV &= dx\,dy\,dz = dx\,dz\,dy \\ &= dy\,dx\,dz = dy\,dz\,dx \\ &= dz\,dx\,dy = dz\,dy\,dx \end{aligned}\right\}$$

Six possible orders

example Use a triple integral to calculate the volume of the solid under $x+2y+3z=6$ in the first octant.

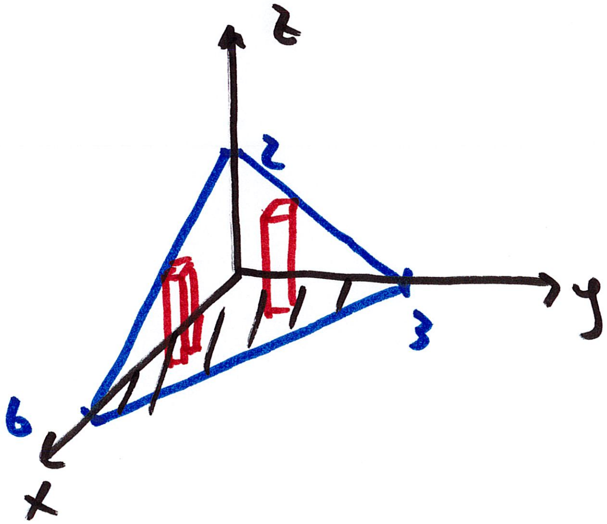
↳ 3D analogy of quadrant



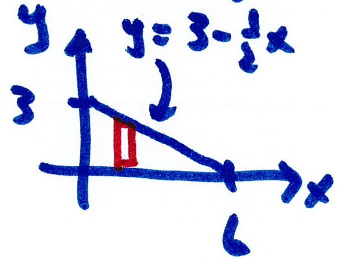
find volume of this prism

volume $= \iiint_D dV$ $\quad (f(x,y,z)=1)$

(just like area $= \iint_R dA$)



look at projection onto the xy-plane



describe this as Type I or II

$0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2}x$

think of this as the "floor" of a room you want to accumulate the height ~~with th~~ within this "floor" area

how high are we allowed to go? $\quad 0 \le z \le 2 - \frac{1}{3}x - \frac{2}{3}y$

from plane eq

bounds: $0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2}x$

$0 \le z \le 2 - \frac{1}{3}x - \frac{2}{3}y$

Basic rule: integrate the variable w/ constant bounds LAST (outside)

integrate the variable w/ most number of variables FIRST

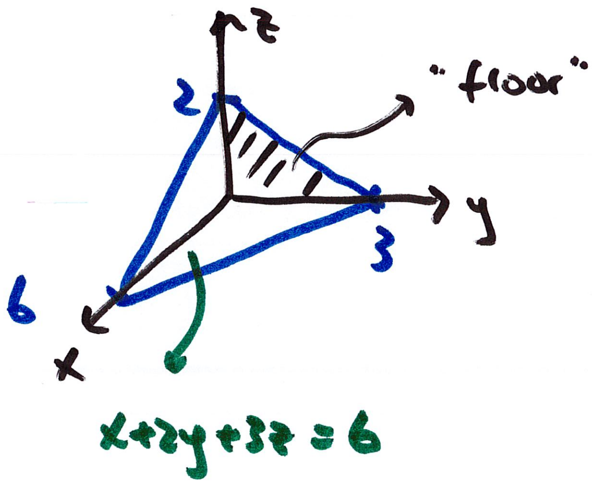
here, x last, z first

volume of this prism $\displaystyle \underset{x}{\int_0^6} \underset{y}{\int_0^{3-\frac{1}{2}x}} \underset{z}{\int_0^{2-\frac{1}{3}x-\frac{2}{3}y}} 1 \cdot \overbrace{dz\,dy\,dx}^{dv}$

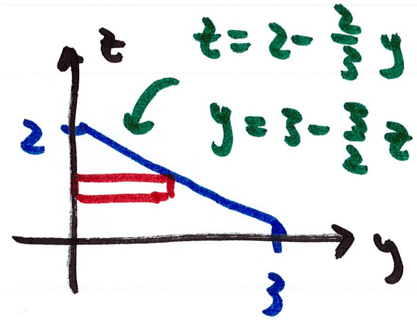
$$= \int_0^6 \int_0^{3-\frac{1}{2}x} z \Big|_{z=0}^{z=2-\frac{1}{3}x-\frac{2}{3}y} dy\,dx = \int_0^6 \int_0^{3-\frac{1}{2}x} \left(2 - \frac{1}{3}x - \frac{2}{3}y\right) dy\,dx$$

$$= \cdots = \boxed{6}$$

let's try a different plane as the "floor"



yz-plane



choice of Type I or II

$$\boxed{0 \le z \le 2}$$

$$\boxed{0 \le y \le 3 - \frac{3}{2}z}$$

"ceiling" is how far out in x we can go

yz-plane $\rightarrow$ $\boxed{0 \le x \le 6 - 2y - 3z}$

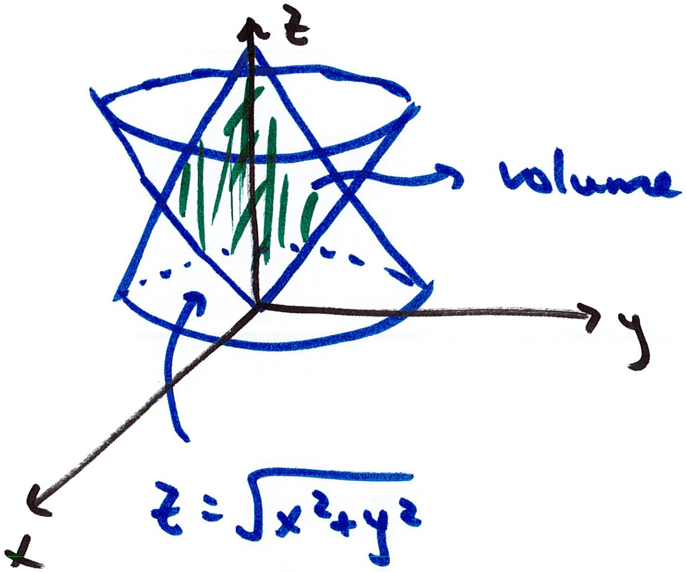
order: x first (has two other variables in bounds)

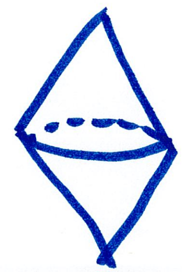
z last (constant bounds)

$$\int_0^2 \int_0^{3-\frac{3}{2}z} \int_0^{6-2y-3z} \overbrace{dx\,dy\,dz}^{dV} = \cdots = \boxed{6}$$

$$z \qquad y \qquad x$$

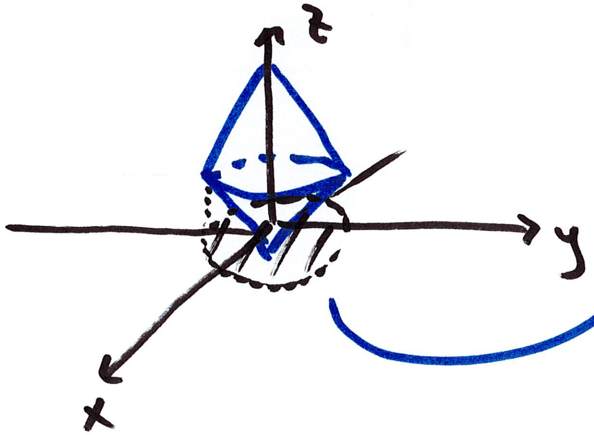
example Find volume above $z = \sqrt{x^2+y^2}$ and below $z = 2 - \sqrt{x^2+y^2}$

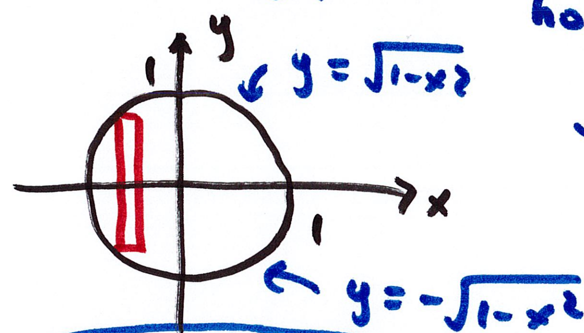


volume of this  ?

pick a plane to project the shape onto

let's pick xy-plane





$\boxed{-1 \le x \le 1}$

$\boxed{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}}$

how big is this circle?

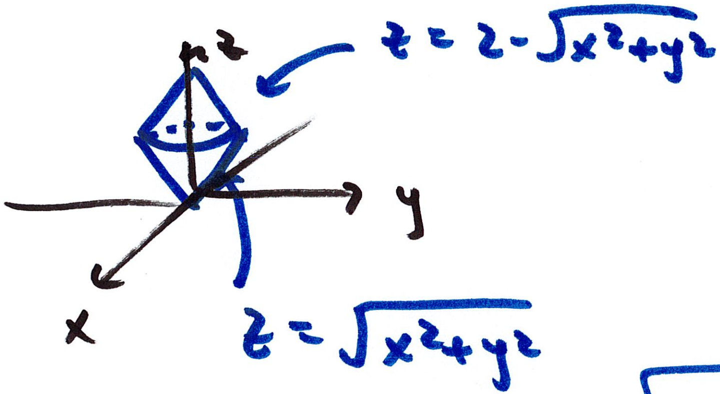
$\sqrt{x^2+y^2} = 2 - \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} = 1 \rightarrow x^2+y^2 = 1$

radius 1

$y = \pm\sqrt{1-x^2}$

$-1 \le x \le 1$

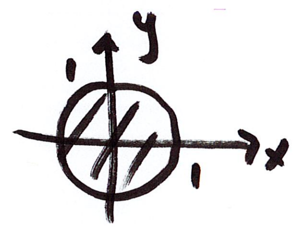


z bounds are the highest and lowest z can be

$$\boxed{\sqrt{x^2+y^2} \le z \le 2-\sqrt{x^2+y^2}}$$

$$\text{volume} = \underset{x}{\int_{-1}^{1}} \underset{y}{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}} \underset{z}{\int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}}} \overbrace{dz\,dy\,dx}^{dv}$$

terrible in Cartesian, go to polar



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$dy\,dx = \underline{\underline{r}}\,dr\,d\theta$

$$\underset{\theta}{\int_0^{2\pi}} \underset{r}{\int_0^{1}} \underset{z}{\int_r^{2-r}} \overbrace{\underline{\underline{r}}\,dz\,dr\,d\theta}^{dv} = \dots = \boxed{\frac{2\pi}{3}}$$

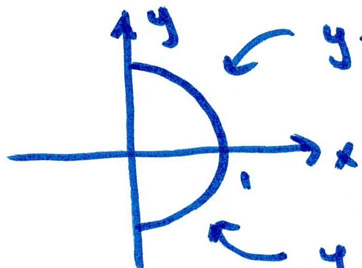
example Rewrite $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$

as $\iiint_D dx\,dz\,dy$ and evaluate

as given $0 \le x \le 1$
$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ } "floor" (last two rounds of integration)
$0 \le z \le \sqrt{1-x^2-y^2}$ "ceiling" first round

Sketch the solid involved: $z = \sqrt{1-x^2-y^2} \leftrightarrow x^2+y^2+z^2 = 1$
sphere
$z = 0$ lower bound $\to$ top hemisphere only

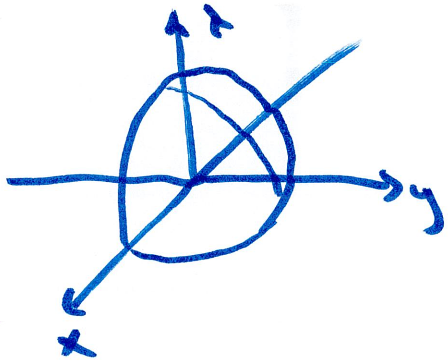
the "floor" is the shadow of the solid



$y = \sqrt{1-x^2}$

$y = -\sqrt{1-x^2}$

this tells us we are looking at the right half of the upper hemisphere of radius 1

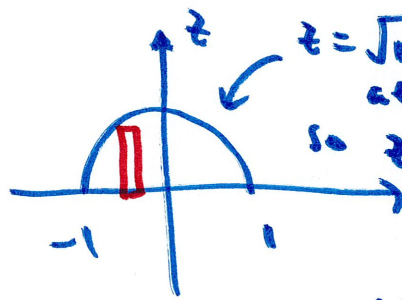


so, even w/o doing any calculation, we already know the volume: $\frac{1}{4} \cdot \frac{4}{3}\pi(1)^3 = \frac{\pi}{3}$

now rewrite order w/ ~~dzdy~~

$dx\,\underbrace{dz\,dy}_{\text{"floor"}}$

y is left so has constant bounds



$z = \sqrt{1-x^2-y^2}$
at $x=0$
so $z = \sqrt{1-y^2}$

$-1 \le y \le 1$

$0 \le z \le \sqrt{1-y^2}$

the "ceiling" is how far we can go in x

$$\underset{\text{yz-plane}}{0} \le x \le \underbrace{\sqrt{1-y^2-z^2}}_{\text{from } z=\sqrt{1-x^2-y^2}}$$

new integral : $\displaystyle \underbrace{\int_{-1}^{1}}_{y} \underbrace{\int_{0}^{\sqrt{1-y^2}}}_{z} \underbrace{\int_{0}^{\sqrt{1-y^2-z^2}}}_{x} dx\,dz\,dy = \frac{\pi}{3}$ (from geometry)