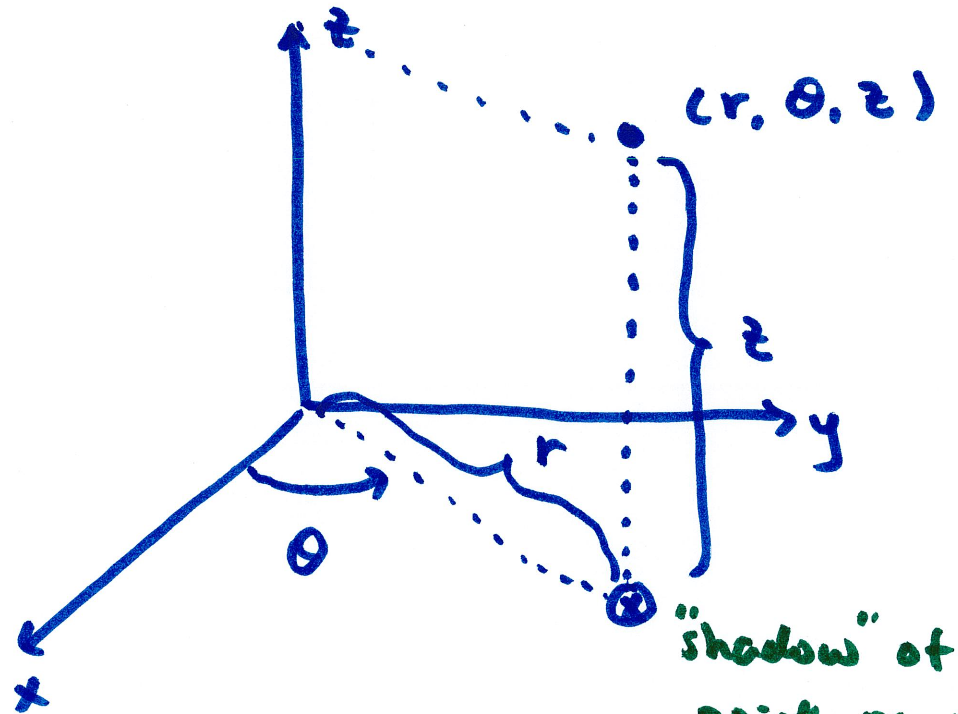
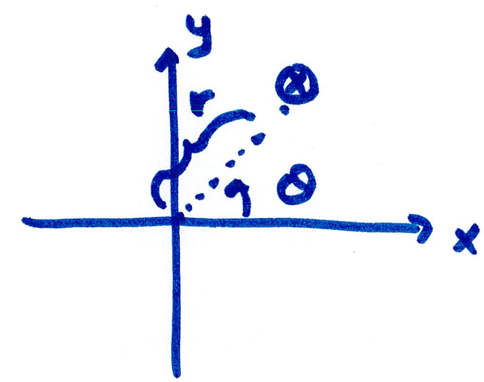
16.5 Triple Integrals in Cylindrical Coordinates

think of cylindrical as a hybrid polar and Cartesian

point in cylindrical : $(\underbrace{r, \theta}_{\text{polar}}, \underbrace{z}_{\text{Cartesian}})$





shadow on xy-plane exactly like polar then lifted up/down by z to locate the point

conversion: $(x, y, z) \to (r, \theta, z)$

$r^2 = x^2 + y^2$
& $\tan\theta = \dfrac{y}{x}$ } same as in polar

$z = z$ — z stays the same

$(r, \theta, z) \to (x, y, z)$

$x = r\cos\theta$
$y = r\sin\theta$ } like polar

$z = z$ — z stays the same

example

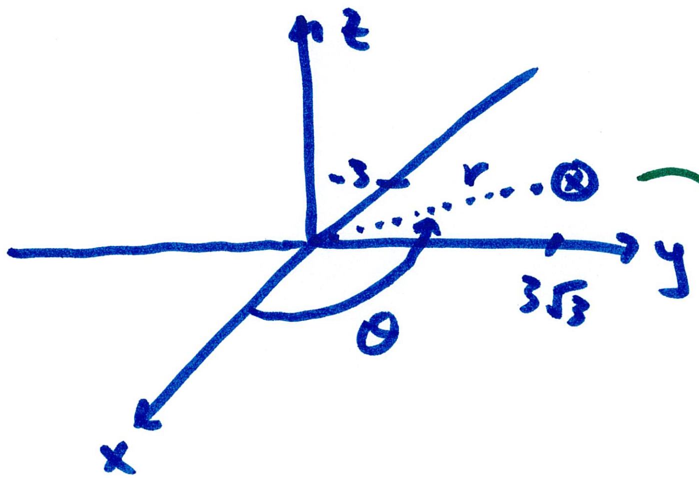
$(x, y, z) = (-3, 3\sqrt{3}, 1)$
$(r, \theta, z) = ?$

find r: $r^2 = x^2 + y^2$
$= 9 + 27 = 36$
$r = 6$ (or $r = -6$)

here, let's use $\boxed{r = 6}$
then θ is in QII

$\tan\theta = \dfrac{y}{x} = \dfrac{3\sqrt{3}}{-3} = -\dfrac{\sqrt{3}}{1}$

$= -\dfrac{\sqrt{3}/2}{1/2} \to \boxed{\theta = 2\pi/3}$

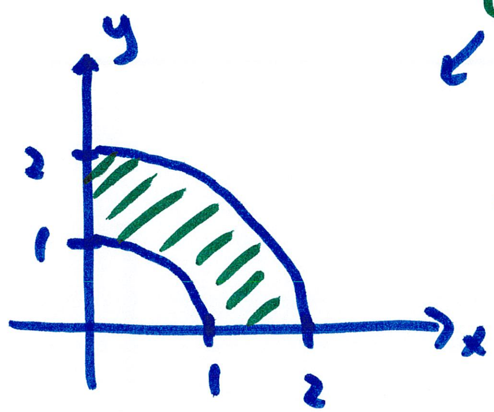


shadow of point on xy-plane

point in cylindrical: $(6, 2\pi/3, 1)$

__example__ sketch the space described by

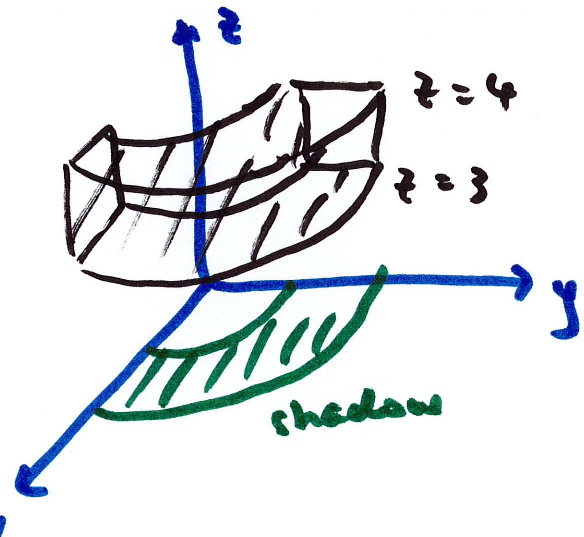
$$\left\{(r,\theta,z) : \underbrace{1 \le r \le 2, \quad 0 \le \theta \le \pi/2}_{\substack{\text{shadow of space on xy-plane} \\ \text{(start w/ this)}}}, \quad 3 \le z \le 4\right\}$$



this is the shadow of space on xy-plane

lift base to $z=3$

stop at $z=4$



cylindrical is good if the space we integrate in is cylinder or cylinder-like

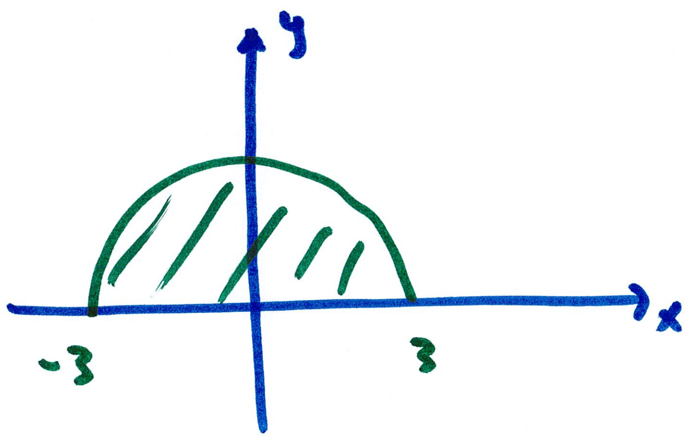
example $\displaystyle\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx$

terrible in Cartesian

the y bounds suggest the shadow of space is circle-like

$$\left.\begin{array}{l} -3 \le x \le 3 \\ 0 \le y \le \underbrace{\sqrt{9-x^2}} \end{array}\right\} \text{shadow of volume}$$

upper half circle radius 3



describe in polar

$0 \le r \le 3$

$0 \le \theta \le \pi$

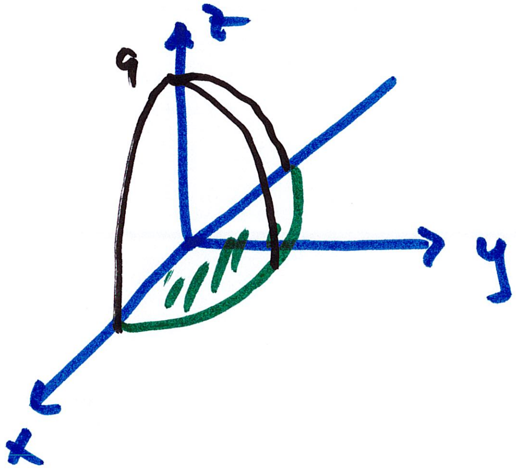
keep z but get rid of x, y

$0 \le z \le 9 - x^2 - y^2$

$9 - x^2 - y^2 = 9 - (x^2 + y^2) = 9 - r^2$

becomes $0 \le z \le 9 - r^2$

paraboloid opening down
vertex at $z = 9$



original integral
$$\underbrace{\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}}}_{\text{replace w/ } r,\ \theta \text{ bounds}} \int_{0}^{\overbrace{9-x^2-y^2}^{9-r^2}} \sqrt{x^2+y^2}\, dz\, \underbrace{dy\, dx}_{\substack{\text{polar equivalent} \\ r\,dr\,d\theta}}$$

turns into r

$$= \int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r \quad r\, dz\, dr\, d\theta$$

$\theta \qquad r \qquad z$

$$= \int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r^2\, dz\, dr\, d\theta = \cdots = \boxed{\frac{162\pi}{5}}$$

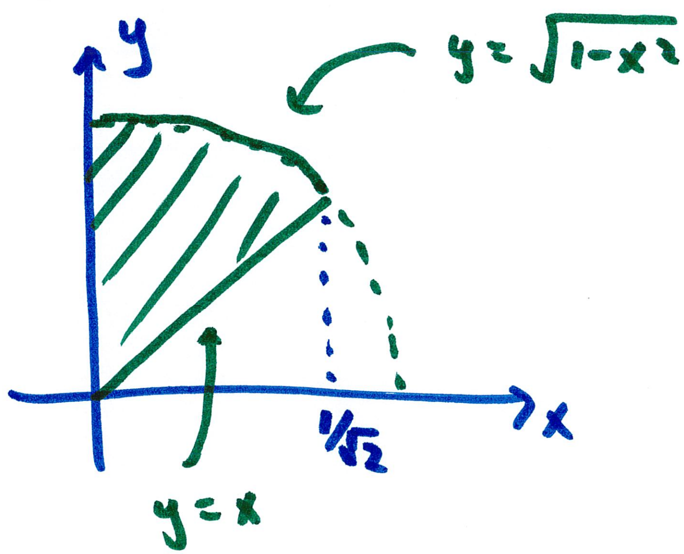
example $\displaystyle\int_0^4 \int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}} e^{-x^2-y^2}\, dy\, dx\, dz$

can't even integrate the first one

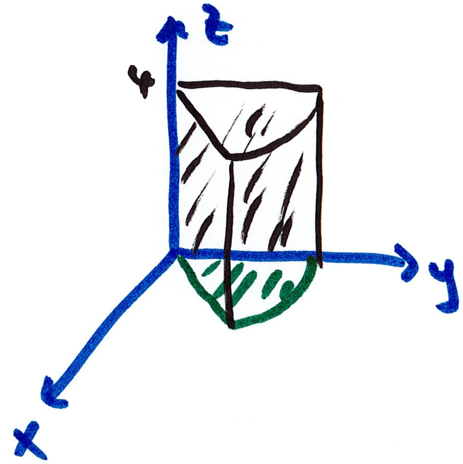
look at bounds of x and y (shadow of space)

$0 \le x \le 1/\sqrt{2}$

$\underset{\text{line}}{x} \le y \le \underbrace{\sqrt{1-x^2}}_{\text{upper half circle radius 1}}$



$0 \le z \le 4$



bounds for the shadow in polar: $0 \le r \le 1$

$\frac{\pi}{4} \le \theta \le \pi/2$

↳ $y = x$ has slope of 1

bisects the QI ($\pi/2$)

so starts at $\pi/4$

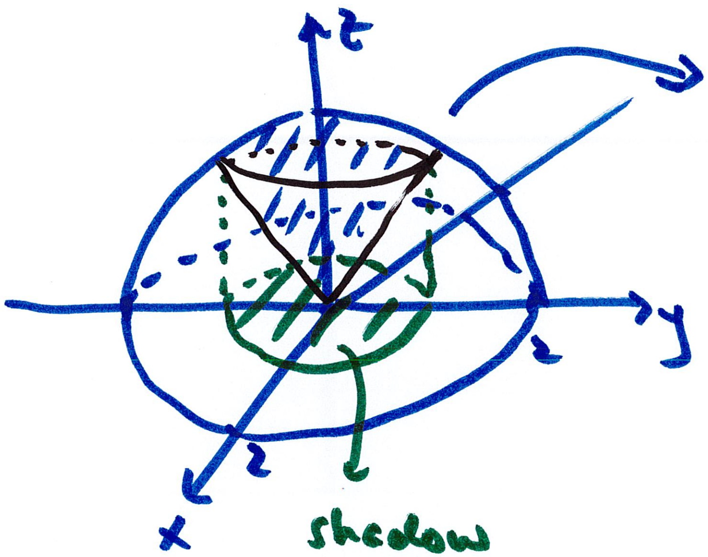
z bounds: $0 \le z \le 4$

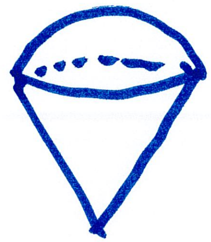
$$\int_0^4 \underbrace{\int_0^{1/\sqrt{2}} \int_x^{\sqrt{1-x^2}}}_{\text{replaced w/ } r, \theta \text{ bounds}} e^{-\overbrace{(x^2+y^2)}^{r^2}} \underbrace{dy\,dx}_{r\,dr\,d\theta}\,dz$$

$$= \int_{0}^{4} \int_{\pi/4}^{\pi/2} \int_{0}^{1} \underbrace{e^{-r^2} \, r\,dr}_{\text{subs: } u = r^2,\ du = 2r\,dr} \, d\theta \, dz = \cdots = \boxed{\frac{\pi}{2}(1-e^{-1})}$$

$z \quad \theta \quad r$

example Find the mass of the solid bounded above by $\underbrace{x^2+y^2+z^2=4}_{\text{sphere radius 2}}$ and bounded below by $\underbrace{z=\sqrt{x^2+y^2}}_{\text{cone}}$ with density $\rho(x,y,z)=z$





mass = ?

shadow circle radius ?

things involve circle or circle-like shapes so stay away from Cartesian

→ to cylindrical

sphere: $z=\sqrt{4-x^2-y^2}$

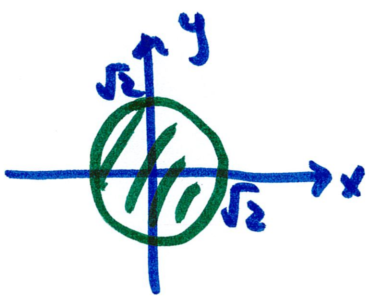
cone: $z=\sqrt{x^2+y^2}$

intersection: same z

$$\sqrt{4-x^2-y^2}=\sqrt{x^2+y^2}$$

$$4-x^2-y^2=x^2+y^2$$

$x^2+y^2=2$ circle radius $\sqrt{2}$



bounds for r, θ: $0 \le r \le \sqrt{2}$

$0 \le \theta \le 2\pi$

bounds for z: $\underset{\text{cone}}{\sqrt{x^2+y^2}} \le z \le \underset{\text{sphere}}{\sqrt{4-x^2-y^2}}$

$\downarrow$

$r \le z \le \sqrt{4-r^2}$

find mass by integrating density $\rho = z$ (given)

$$\underset{\theta}{\int_0^{2\pi}} \underset{r}{\int_0^{\sqrt{2}}} \underset{z}{\int_r^{\sqrt{4-r^2}}} z \; r\,dz\,dr\,d\theta = \dots = \boxed{2\pi}$$