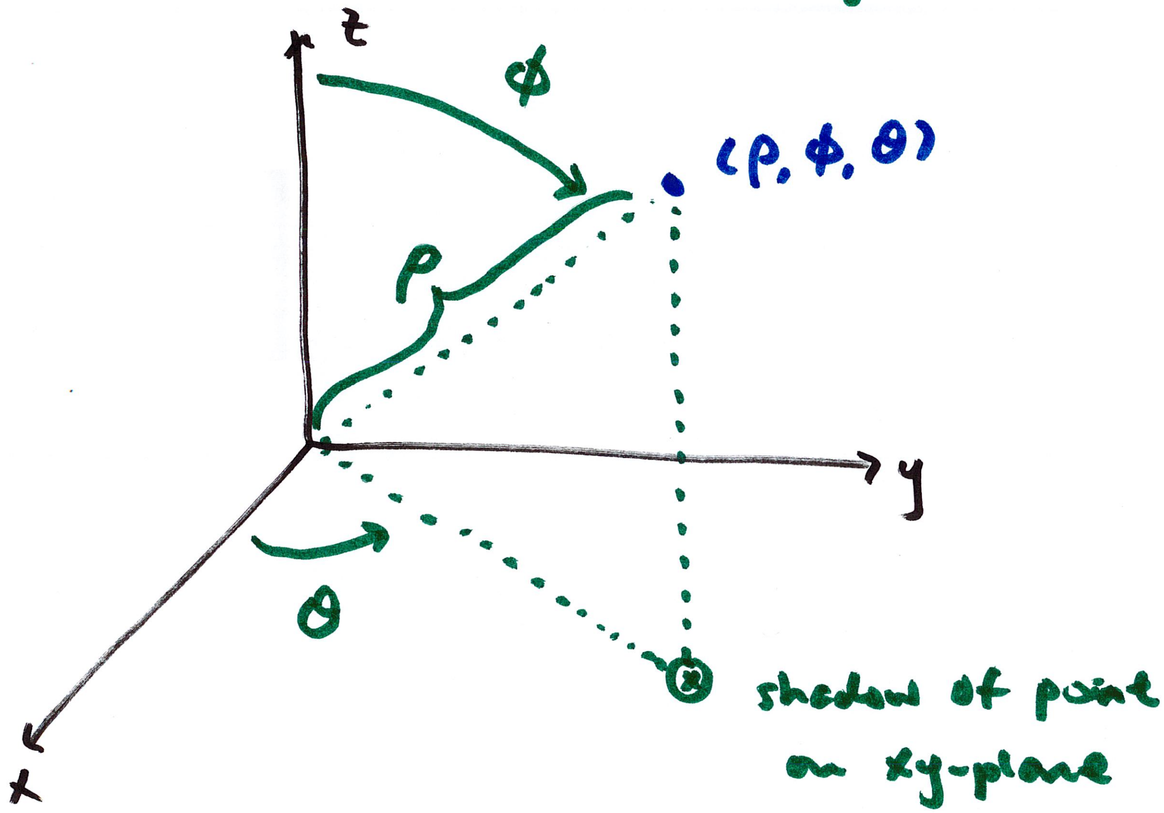
16.5 Triple Integrals in Spherical Coordinates

in spherical, a point is located by (ρ, ϕ, θ)

same θ in polar/cylindrical

"rho" distance from origin

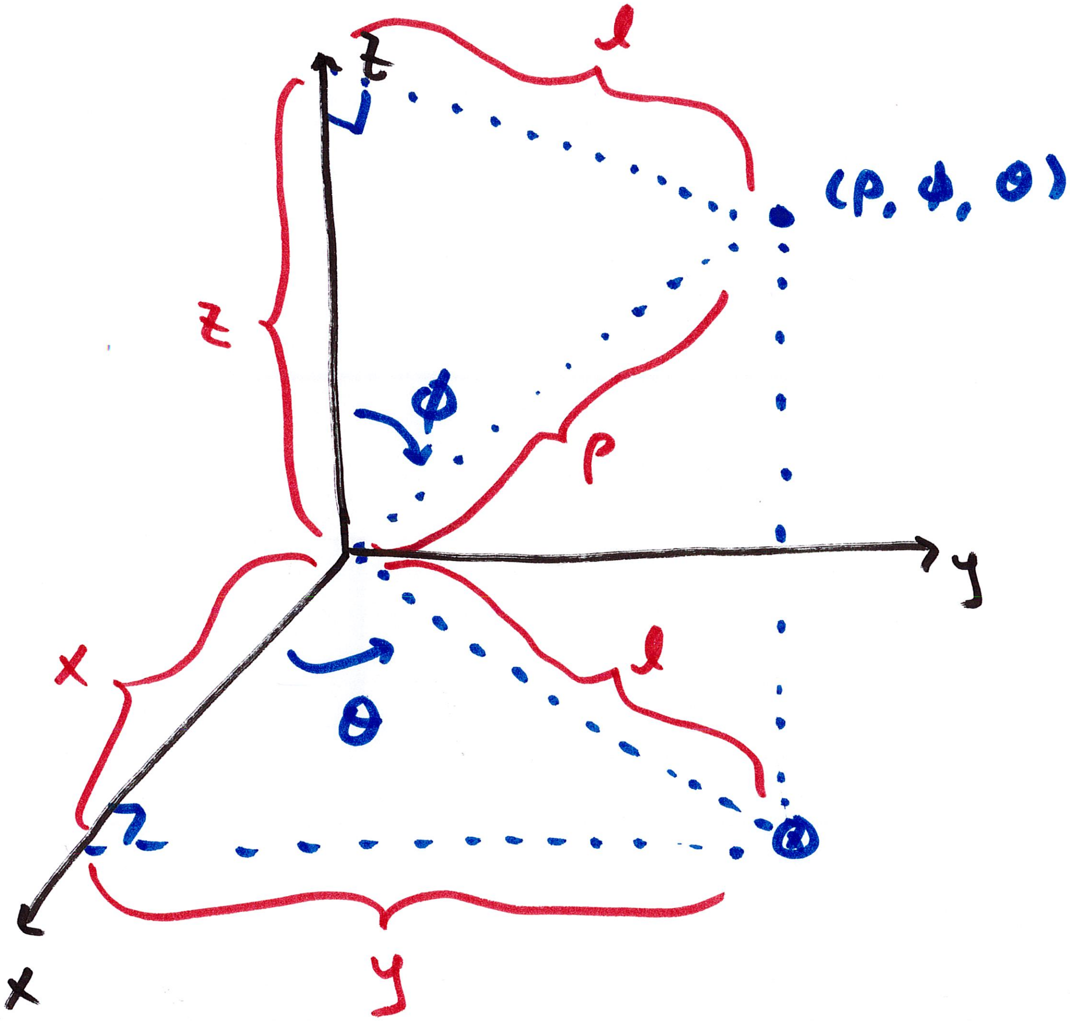
"phi" angle measured from positive z-axis down



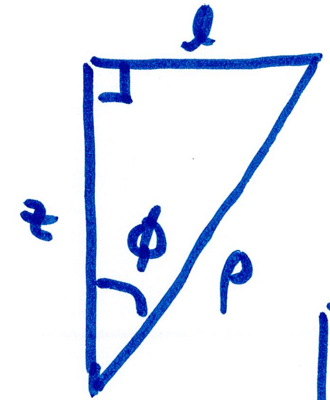
$$\boxed{\begin{array}{l} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{array}}$$

we don't need ϕ to go to 2π because we rely on θ to cover all sides around

Converting to/from Cartesian



top triangle:



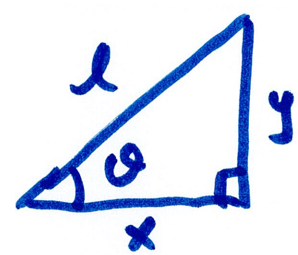
$\cos\phi = \frac{z}{\rho}$

$\boxed{z = \rho\cos\phi}$

also, $\sin\phi = \frac{\ell}{\rho}$

so, $\ell = \rho\sin\phi$

bottom triangle:

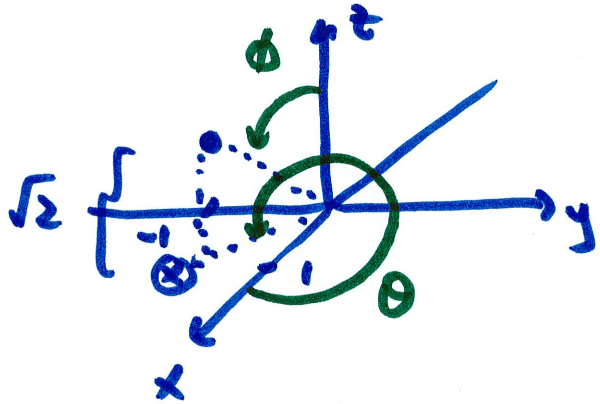


$\cos\theta = \frac{x}{\ell} \rightarrow x = \ell\cos\theta \rightarrow \boxed{x = \rho\sin\phi\cos\theta}$

$\sin\theta = \frac{y}{\ell} \rightarrow y = \ell\sin\theta \rightarrow \boxed{y = \rho\sin\phi\sin\theta}$

note: $\boxed{x^2 + y^2 + z^2 = \rho^2}$

<u>example</u> $(x, y, z) = (1, -1, \sqrt{2})$ $\quad (\rho, \phi, \theta) = ?$



the picture shows $\quad 0 \le \phi \le \pi/2$

$\frac{3\pi}{2} \le \theta \le 2\pi$

ρ is easy: $\rho^2 = x^2+y^2+z^2 = 1+1+2 = 4$ $\quad \boxed{\rho = 2}$

use $z = \rho\cos\phi$ to find ϕ

$$\sqrt{2} = 2\cos\phi \rightarrow \cos\phi = \frac{\sqrt{2}}{2} \rightarrow \boxed{\phi = \pi/4}$$

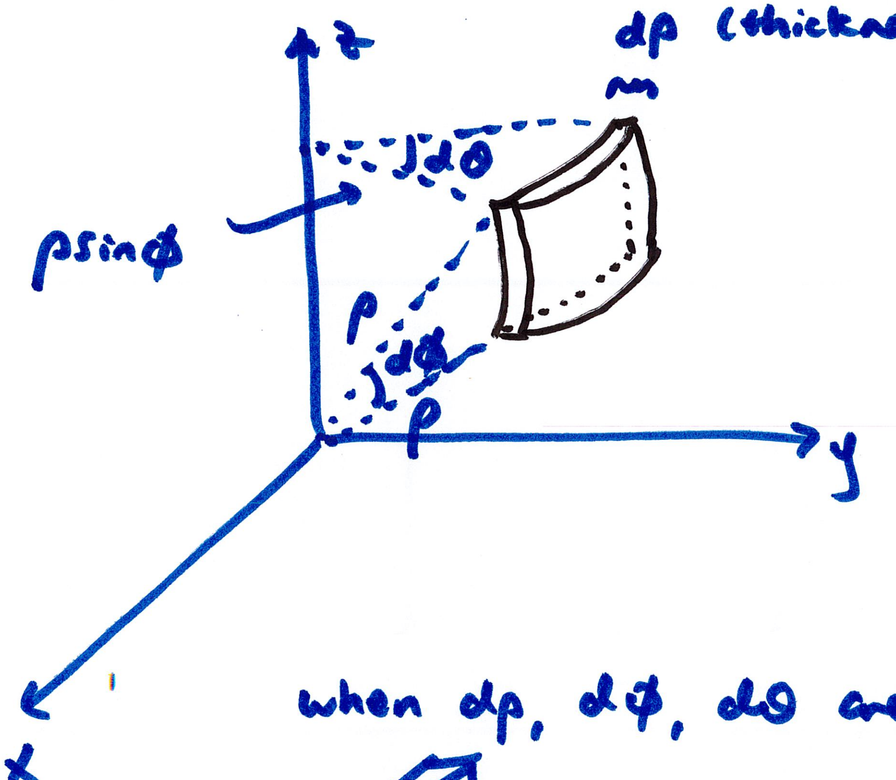
$$\left.\begin{array}{l} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \end{array}\right\} \quad \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{-1}{1} = -1$$

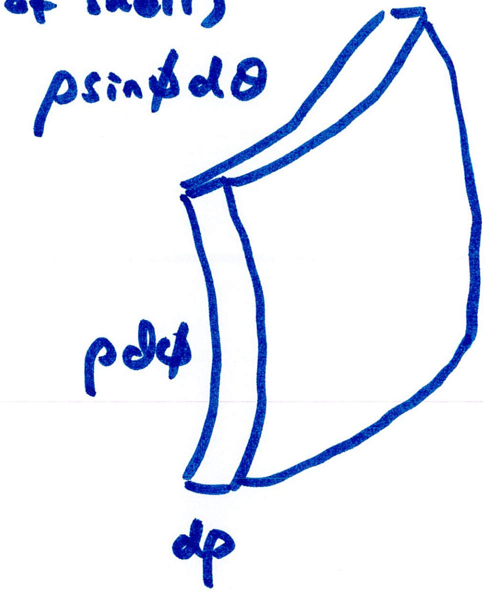
find θ in $\frac{3\pi}{2} \le \theta \le 2\pi$ such that

$$\tan\theta = -1 \rightarrow \boxed{\theta = \frac{7\pi}{4}}$$

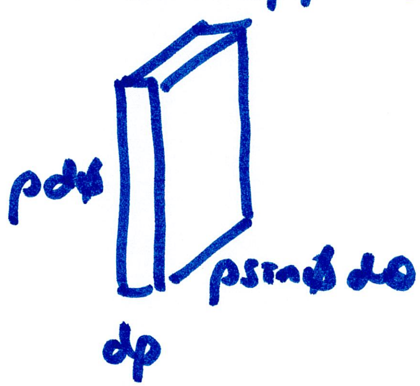
dV in spherical is more complicated

imagine a thin slice of a spherical shell





when $d\rho$, $d\phi$, $d\theta$ are small, the shell $\approx$ rectangle



volume = $\boxed{dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$

example

$$\int_0^6 \int_0^{\sqrt{36-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{72-x^2-y^2}} dz\,dy\,dx$$

terrible integral in Cartesian

the upper bound of z is $\sqrt{72-x^2-y^2}$ which is the upper half of a sphere radius $\sqrt{72}$

so we are working in a sphere-like volume

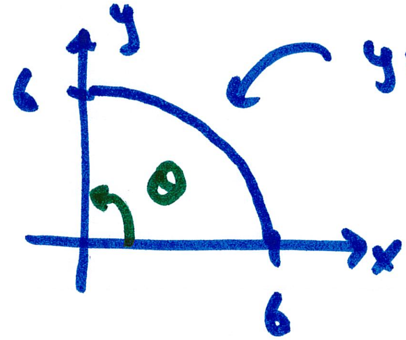
spherical is often a good choice

(box-like : Cartesian, cylinder-like : cylindrical)

sketch the volume from x, y, z bounds and find bounds for ρ, ϕ, θ

$0 \le x \le 6$

$0 \le y \le \sqrt{36-x^2}$

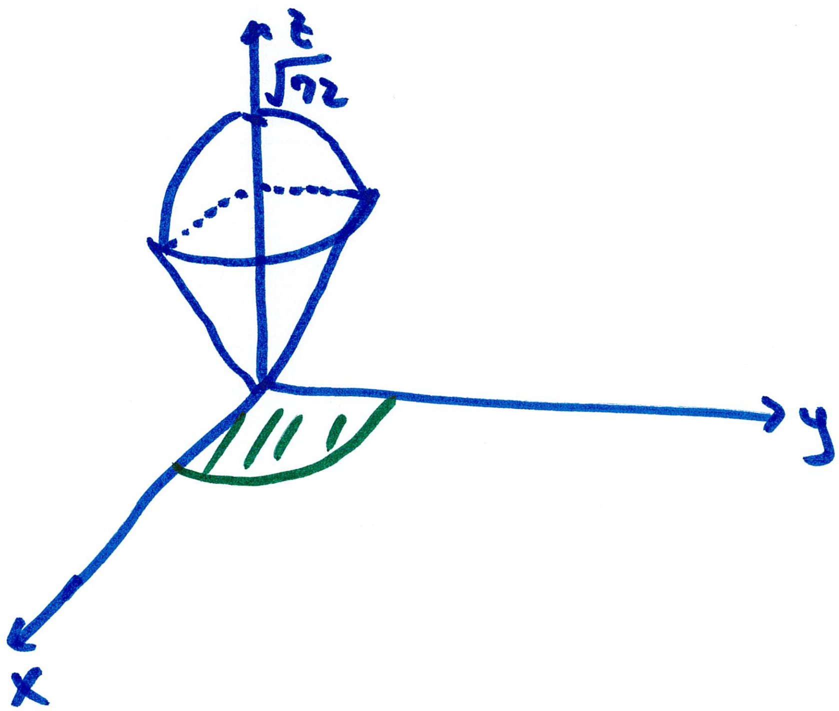


$y = \sqrt{36-x^2}$

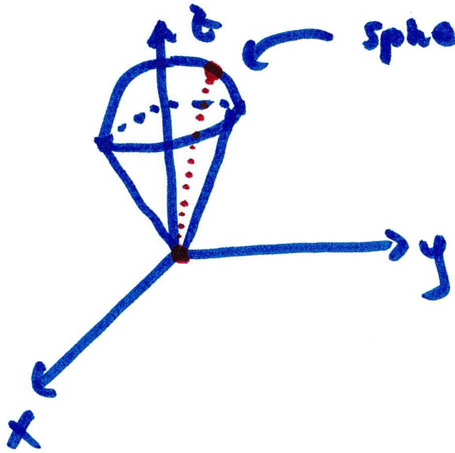
this is the "shadow" of the volume on xy-plane

we see $\boxed{0 \le \theta \le \pi/2}$

$$\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{72-x^2-y^2}}_{\text{sphere radius } \sqrt{72}}$$



upperbound of ρ is the distance from the origin to the edge of volume

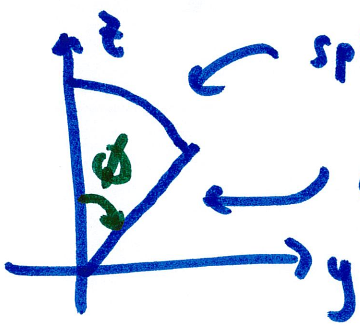


include all points from origin ($\rho = 0$) to the sphere ($\rho = \sqrt{72}$)

$$\boxed{0 \le \rho \le \sqrt{72}}$$

now ϕ bounds : from z-axis to the edge of cone

on ~~yEz~~ yz-plane



cone equation: $z = \sqrt{x^2+y^2}$

on yz-plane: $z = \sqrt{y^2} = y$

so slope is 1 which bisects quadrant, so $\phi = \pi/4$

therefore,

$$\boxed{0 \le \phi \le \pi/4}$$

now the integral:

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sqrt{72}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV \text{ (used to be } dz\,dy\,dx)} = \cdots = \boxed{12\sqrt{72}\,\pi\left(1-\frac{1}{\sqrt{2}}\right)}$$

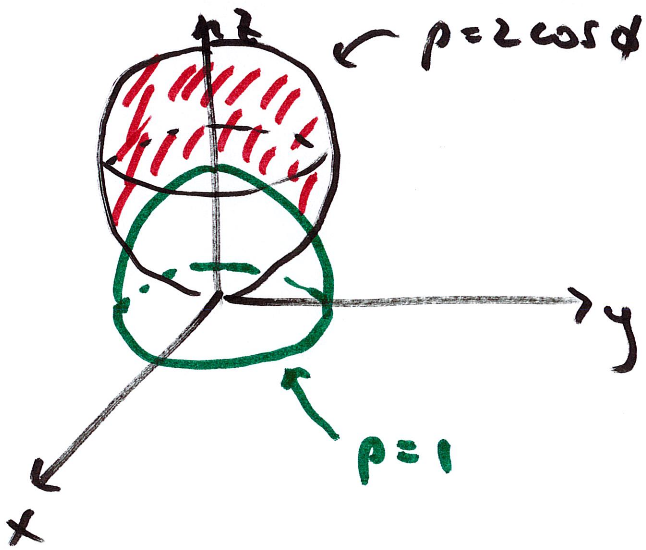
example Find volume of solid outside $\underbrace{\rho=1}_{\text{sphere radius 1 centered }(0,0,0)}$ and inside $\underbrace{\rho=2\cos\phi}_{?}$

what is $\rho=2\cos\phi$?

$$\sqrt{x^2+y^2+z^2} = \frac{2\overbrace{\rho\cos\phi}^{z}}{\rho} = \frac{2z}{\sqrt{x^2+y^2+z^2}}$$

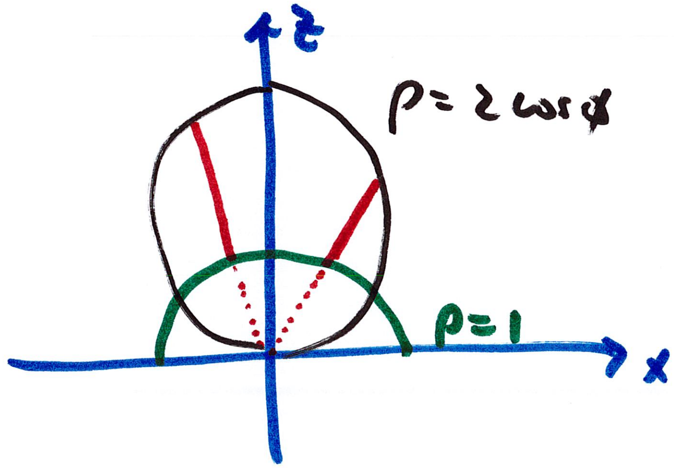
$$x^2+y^2+z^2=2z \longrightarrow x^2+y^2+(z-1)^2=1$$

sphere radius 1 center (0, 0, 1)



just by looking at this, we know $\boxed{0\le\theta\le 2\pi}$

ρ bounds: project onto xz-plane

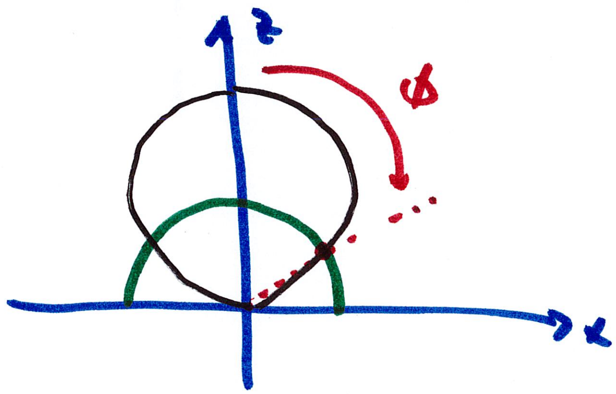


accumulate ρ from $\rho = 1$

up to $\rho = 2\cos\phi$

so, $\boxed{1 \le \rho \le 2\cos\phi}$

ϕ bounds: from z-axis to intersection of spheres



intersection of $\rho = 1$ and $\rho = 2\cos\phi$

$1 = 2\cos\phi$

$\cos\phi = \frac{1}{2} \rightarrow \phi = \pi/3$

(must be between 0 and $\pi/2$ because above xy-plane)

so, $\boxed{0 \le \phi \le \pi/3}$

volume: $\displaystyle\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=1}^{2\cos\phi} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} = \dots = \boxed{\frac{11\pi}{2}}$