16.6 Integrals for Mass Calculations

mass of things in 2D: $\iint\limits_R \rho(x,y)\,dA$ density

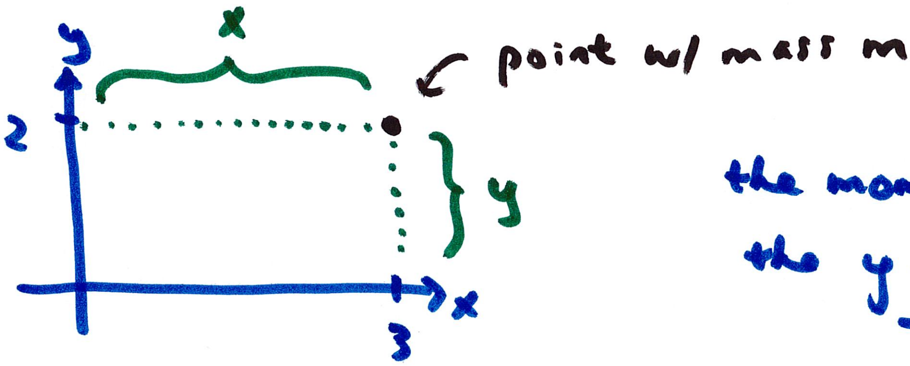
3D: $\iiint\limits_D \rho(x,y,z)\,dV$

we want to calculate the center of mass of objects

first we need to know how to calculate the moment

(specifically the mass moment)
rotational equivalent of momentum

mass moment = mass · distance from rotational axis



the moment of the point mass about the y-axis is

$$M_y = \text{mass} \cdot \underbrace{\text{distance from axis of rotation}}_{x}$$

$$M_y = m x$$

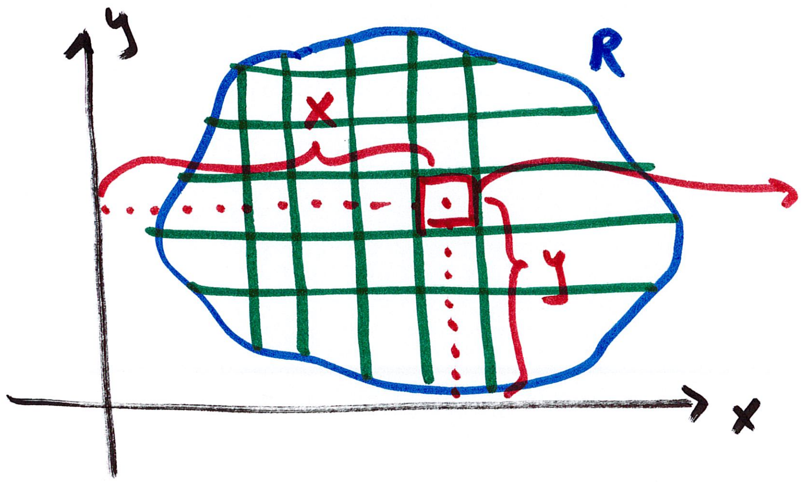
likewise, the moment about x-axis is $M_x = m y$

for example, if $m = 4$

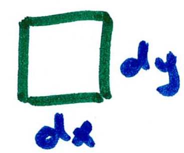
then for this point mass, $M_y = 4 \cdot 3 = 12$

$$M_x = 4 \cdot 2 = 8$$

we will use the point mass idea to find the moments of a region



chop into little pieces



this piece has mass

$$m = \rho(x,y)\,dx\,dy$$

$$m = \rho(x,y)\,dA$$

the small piece is at distance x from y-axis

so $\quad M_y = \rho(x,y)\cdot x\,dA$

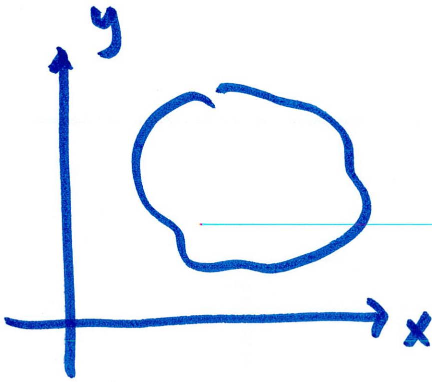
and $\quad M_x = \rho(x,y)\cdot y\,dA$

to find moments of entire region (little pieces have varying x, y, ρ) we accumulate by integration

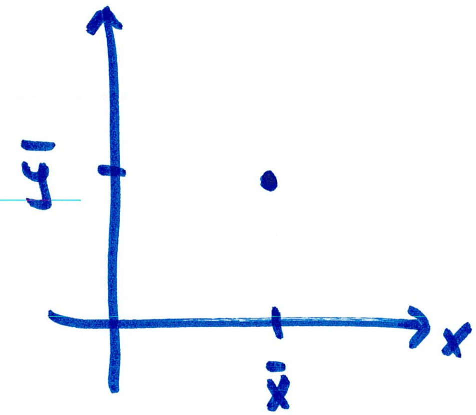
$$\boxed{M_y = \iint_R x\rho\,dA} \qquad \boxed{M_x = \iint_R y\rho\,dA}$$

"x bar" "y bar"

One definition of center of mass: the location $(\bar{x}, \bar{y})$ where a point with same mass as region would be if it were to have the same moments.



collapse into one point $\longrightarrow$



has $M_x = \iint_R y \rho \, dA$

$M_y = \iint_R x \rho \, dA$

mass $= \iint_R \rho \, dA$

want same M_x, M_y

$M_x = m\bar{y}$

$M_y = m\bar{x}$

equate the moments, solve for $\bar{x}$, $\bar{y}$

$$M_x = \iint_R y\rho\, dA = m\bar{y}$$

$$M_y = \iint_R x\rho\, dA = m\bar{x}$$

we get

$$\cancel{\bar{x} = \frac{1}{m} \iint_R \cancel{y}\overset{x}{}\rho\, dA}$$

$$\cancel{\bar{y} = \frac{1}{m} \iint_R y\rho\, dA}$$

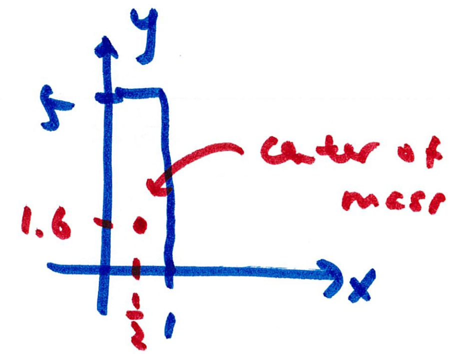
~~where~~ $\cancel{m = \iint_R \rho\, dA}$

$$\boxed{\begin{aligned} \bar{x} &= \frac{1}{m} \iint_R x\rho\, dA \\ \bar{y} &= \frac{1}{m} \iint_R y\rho\, dA \\ m &= \iint_R \rho\, dA \end{aligned}}$$

example

$$R = \{(x,y) : \ 0 \le x \le 1, \quad 0 \le y \le 5\}$$

$$\rho(x,y) = 2e^{-\frac{1}{2}y}$$



first, mass : $m = \iint_R \rho(x,y)\,dA = \int_0^1 \int_0^5 2e^{-\frac{1}{2}y}\,dy\,dx = \cdots = \cancel{4(1-e^{-5/2})}$

$$= 4(1-e^{-5/2})$$

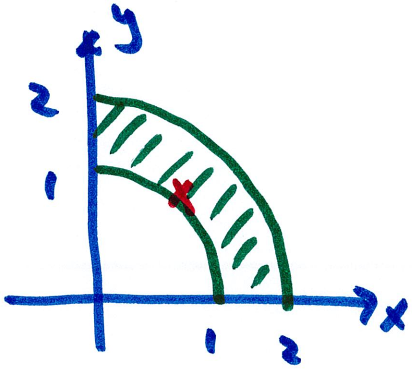
$$\bar{x} = \frac{1}{m}\iint_R x\rho\,dA = \frac{1}{4(1-e^{-5/2})}\int_0^1 \int_0^5 x \cdot 2e^{-\frac{1}{2}y}\,dy\,dx = \cdots = \boxed{\frac{1}{2}}$$

$$\bar{y} = \frac{1}{m}\iint_R y\rho\,dA = \frac{1}{4(1-e^{-5/2})}\int_0^1 \int_0^5 y \cdot 2e^{-\frac{1}{2}y}\,dy\,dx$$

$$= \cdots = \frac{-14e^{-5/2}+4}{2(1-e^{-5/2})} \approx \boxed{1.6}$$

example R: circles of radii 1 and 2 centered at origin in 1st quadrant with density $\rho(x,y) = \underbrace{\sqrt{x^2+y^2}}$

in polar $\sqrt{x^2+y^2} = \sqrt{r^2}$
$= r$



circles, so go to polar

$1 \leq r \leq 2$

$0 \leq \theta \leq \pi/2$

mass: $m = \iint_R \rho \, dA = \int_0^{\pi/2} \int_1^2 \underset{\downarrow \atop \rho}{r} \cdot \overbrace{r\,dr\,d\theta}^{dA} = \cdots = \dfrac{7\pi}{6}$

$\bar{x} = \dfrac{1}{m} \iint_R x \rho \, dA = \dfrac{1}{7\pi/6} \int_0^{\pi/2} \int_1^2 r\cos\theta \cdot \underset{\downarrow \atop \rho}{r} \cdot \underbrace{r\,dr\,d\theta}_{dA} = \dfrac{45}{14\pi} \approx 1.023$

↳ in polar

$x = r\cos\theta$

$\bar{y} = \dfrac{1}{m} \iint_R y \rho \, dA = \dfrac{1}{7\pi/6} \int_0^{\pi/2} \int_1^2 \underbrace{r\sin\theta}_{y} \cdot \underset{\downarrow \atop \rho}{r} \cdot \underbrace{r\,dr\,d\theta}_{dA} = \dfrac{45}{14\pi} \approx 1.023$

↳ $r\sin\theta$

2D to 3D : change dA to dV , $\rho(x,y) = \rho(x,y,z)$

double integral $\rightarrow$ triple

$$m = \iiint_D \rho(x,y,z)\, dv$$

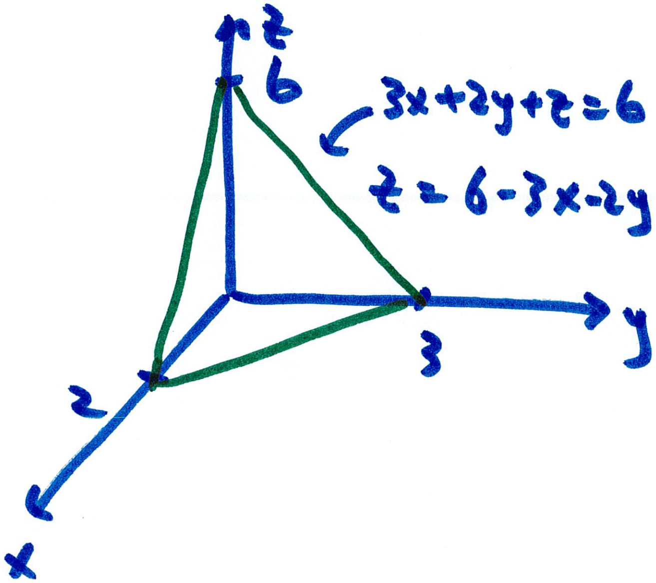
$$\bar{x} = \frac{1}{m} \iiint_D x\, \rho(x,y,z)\, dv$$

$$\bar{y} = \frac{1}{m} \iiint_D y\, \rho(x,y,z)\, dv$$

$$\bar{z} = \frac{1}{m} \iiint_D z\, \rho(x,y,z)\, dv$$

Example Find center of mass of solid bounded by $3x+2y+z=6$ and the coordinate planes.

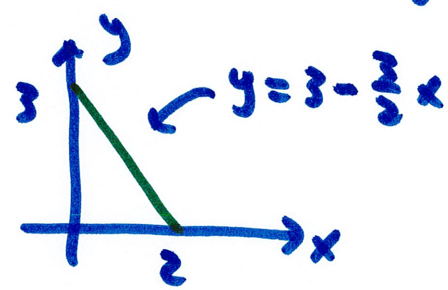
$\rho(x,y,z) = 1+x$



describe the volume D

pick a plane to be the "floor"

here, let's use xy-plane



$$\boxed{\begin{aligned} &0 \le x \le 2 \\ &0 \le y \le 3 - \tfrac{3}{2}x \end{aligned}}$$

z: from xy-plane ($z=0$) to the plane ($z = 6-3x-2y$)

$$\boxed{0 \le z \le 6-3x-2y}$$

mass: $m = \iiint_D \rho \, dV = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} \underbrace{(1+x)}_{\rho} \underbrace{dz\,dy\,dx}_{dV} = \cdots = 9$

$\bar{x} = \frac{1}{m} \iiint_D x\rho \, dV = \frac{1}{9} \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} x\cdot(1+x) \, dz\,dy\,dx = \cdots = 3/5$

Similarly, $\bar{y} = \cdots = 7/10$, $\bar{z} = \cdots = 7/5$

a related quantity, called <u>moment of inertia</u> (or angular mass)

is the rotational equivalent of mass, high → hard to rotate

$$\underbrace{I_{axis}}_{\text{moment of inertia about an axis}} = \iint_R \rho(x,y)\, d^2\, dA \quad (2D)$$

~~net~~ moment of inertia about an axis

d is distance to that axis

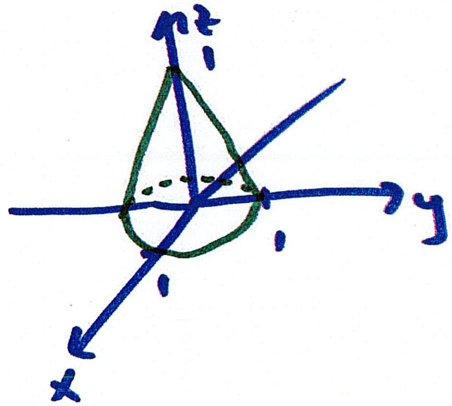
("x" in $M_y = \iint_R x\, dA$)

$$I_{axis} = \iiint_D \rho(x,y,z)\, d^{\cancel{3}\,2}\, dV \quad (3D)$$

example Find the moment of inertia of the solid bounded above by

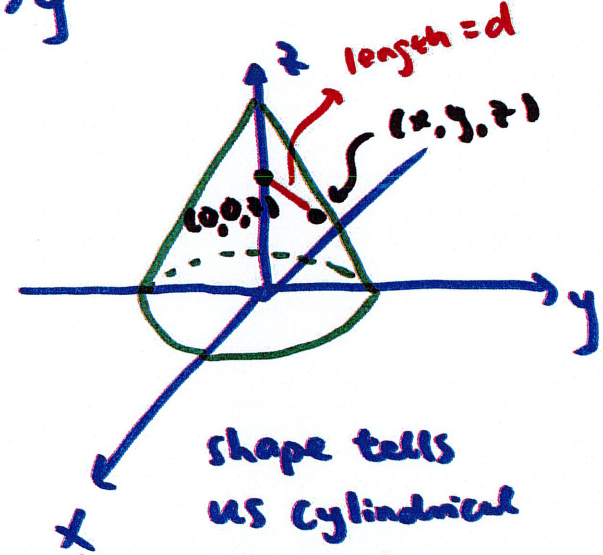
$z = 1 - \sqrt{x^2+y^2}$ and below by $z=0$ about the z-axis

$\rho(x,y,z) = z$



we want $I_z = \iiint\limits_D \rho(x,y,z)\, d^2\, dV$

d: distance of any point in the shape from z-axis



shape tells us cylindrical is good

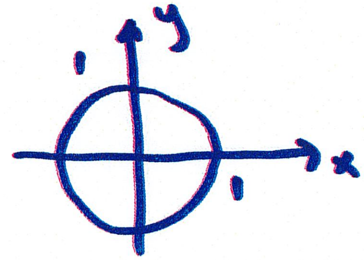
d is distance from (x,y,z) to $(0,0,z)$

so, $d \neq x^2+y^2$

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$

$$= \sqrt{x^2+y^2}$$

use xy-plane as "floor"



$0 \le r \le 1$

$0 \le \theta \le 2\pi$

cylindrical is good, so $0 \le z \le 1-r$

$$I_z = \iiint_D \rho(x,y,z)\, d^2\, dV$$

$$= \int_0^{2\pi}\int_0^1\int_0^{1-r} \underbrace{z}_{\rho}\, \underbrace{(r)}_{d=\sqrt{x^2+y^2}=r}{}^2\, \overbrace{r\,dz\,dr\,d\theta}^{dV} = \cdots = \boxed{\frac{\pi}{60}}$$