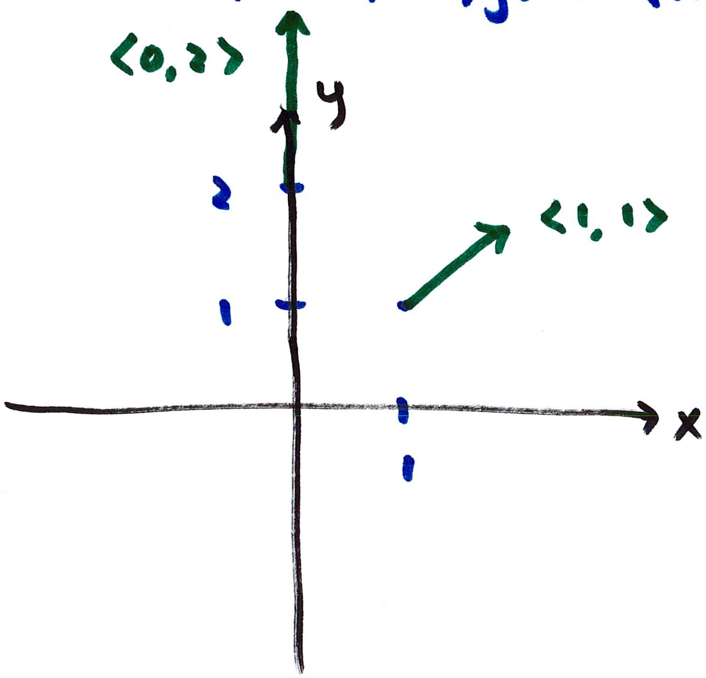
17.1 Vector Fields

a vector field is a function that assigns a <u>vector</u> to a point

for example, $\vec{F}(x,y) = \langle x, y \rangle = x\vec{i} + y\vec{j}$

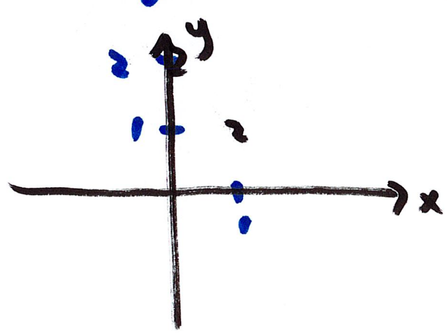


at $(1,1)$ $\vec{F}(1,1) = \langle 1, 1 \rangle$

at $(0,2)$ $\vec{F}(0,2) = \langle 0, 2 \rangle$

in contrast, the functions we have been working with are <u>scalar fields</u> - a function that assigns a scalar to a point

for example, $f(x,y) = x+y$



at $(1,1)$ a $f(1,1)=2$

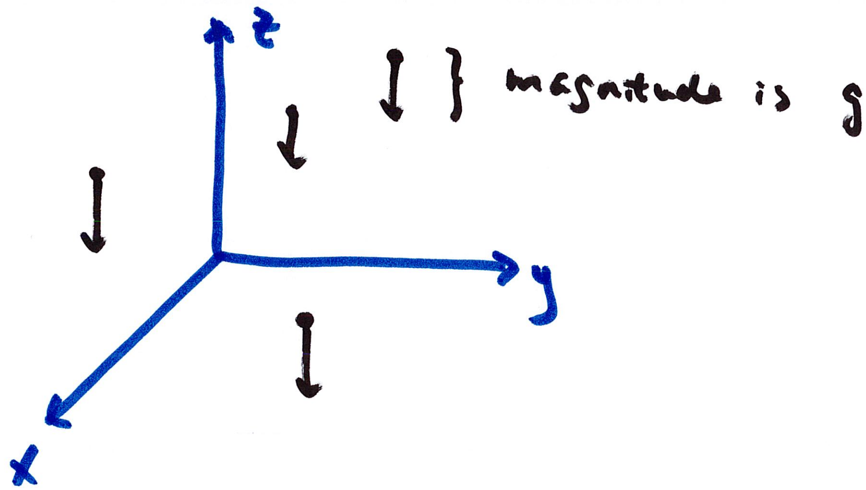
at $(0,2)$ $f(0,2)=2$

the temperature distribution in this room is a scalar field

- each point in the space of this room has a temperature (scalar)

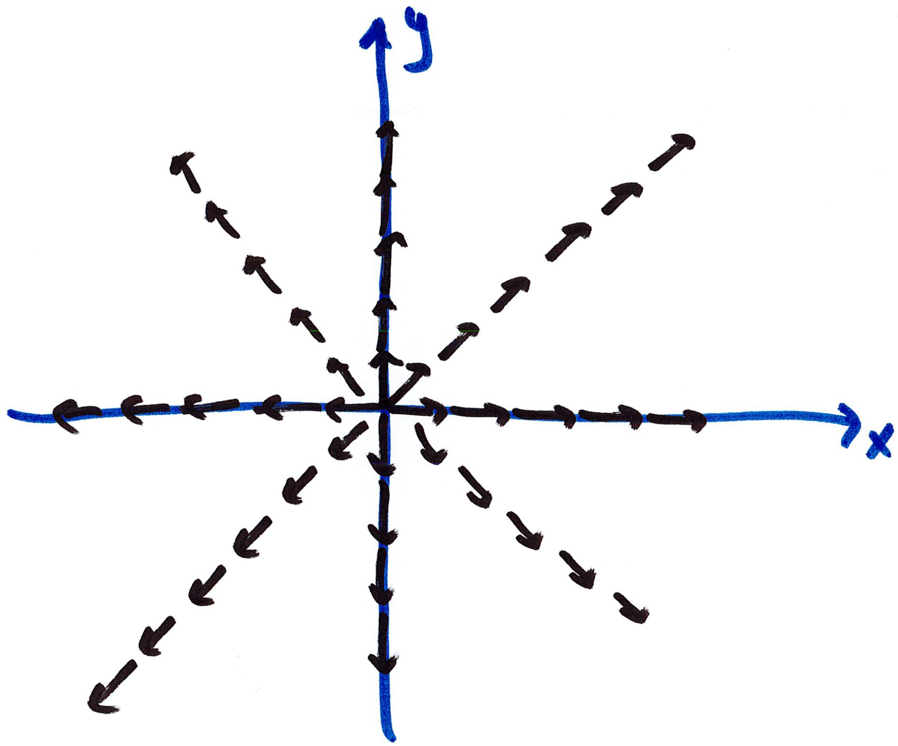
the acceleration due to gravity in this room is a vector field

$\vec{F}(x,y,z) = \langle 0, 0, -g \rangle$



<u>example</u> $\vec{F}(x,y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

we can pick points like in last example
but there is a more efficient way



$$\vec{F} = \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle$$

$$|\vec{F}| = \left| \frac{1}{\sqrt{x^2+y^2}} \right| \left| \langle x, y \rangle \right|$$

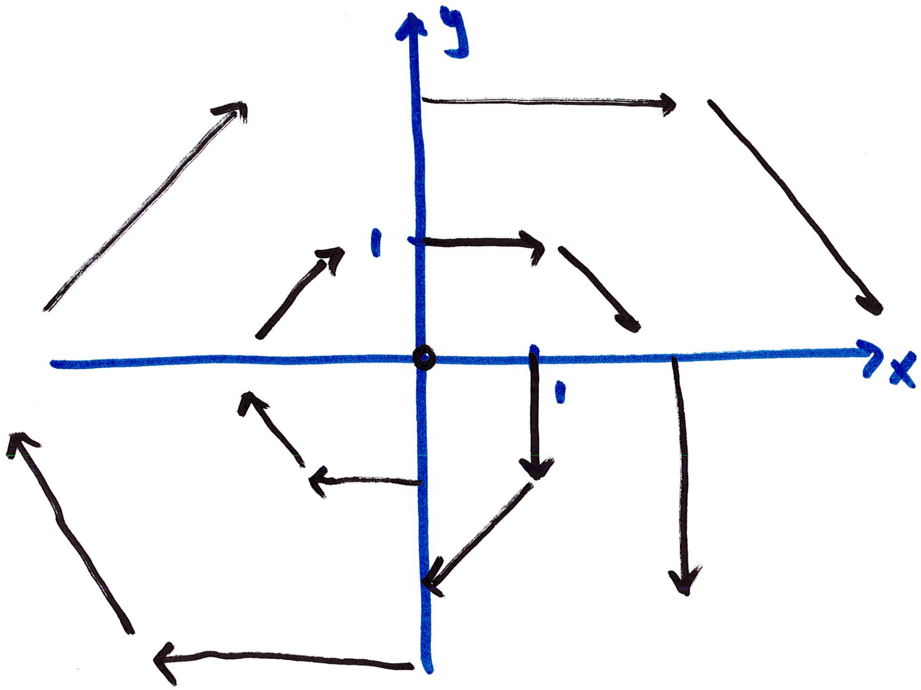
$$= \frac{1}{\sqrt{x^2+y^2}} \sqrt{x^2+y^2} = 1$$

each vector is a unit vector
the direction is given by $\langle x, y \rangle$
which is radially away from origin

this vector field has a domain
that does NOT include origin

let's look at a few examples of sketching vector fields

<u>example</u> $\vec{F}(x,y) = \langle y, -x \rangle$



$\vec{F}(0,0) = \langle 0,0 \rangle = \vec{0}$ (zero vector)

$\vec{F}(1,0) = \langle 0,-1 \rangle$

$\vec{F}(0,1) = \langle 1, 0 \rangle$

$\vec{F}(0,-1) = \langle -1, 0 \rangle$

$\vec{F}(1,-1) = \langle -1, -1 \rangle$

$\vec{F}(2,0) = \langle 0, -2 \rangle$

clockwise spiral with larger magnitude away from origin if the vector field $\vec{F}$ is the gradient of some scalar field then we call the scalar field the <u>potential function</u> of the vector field

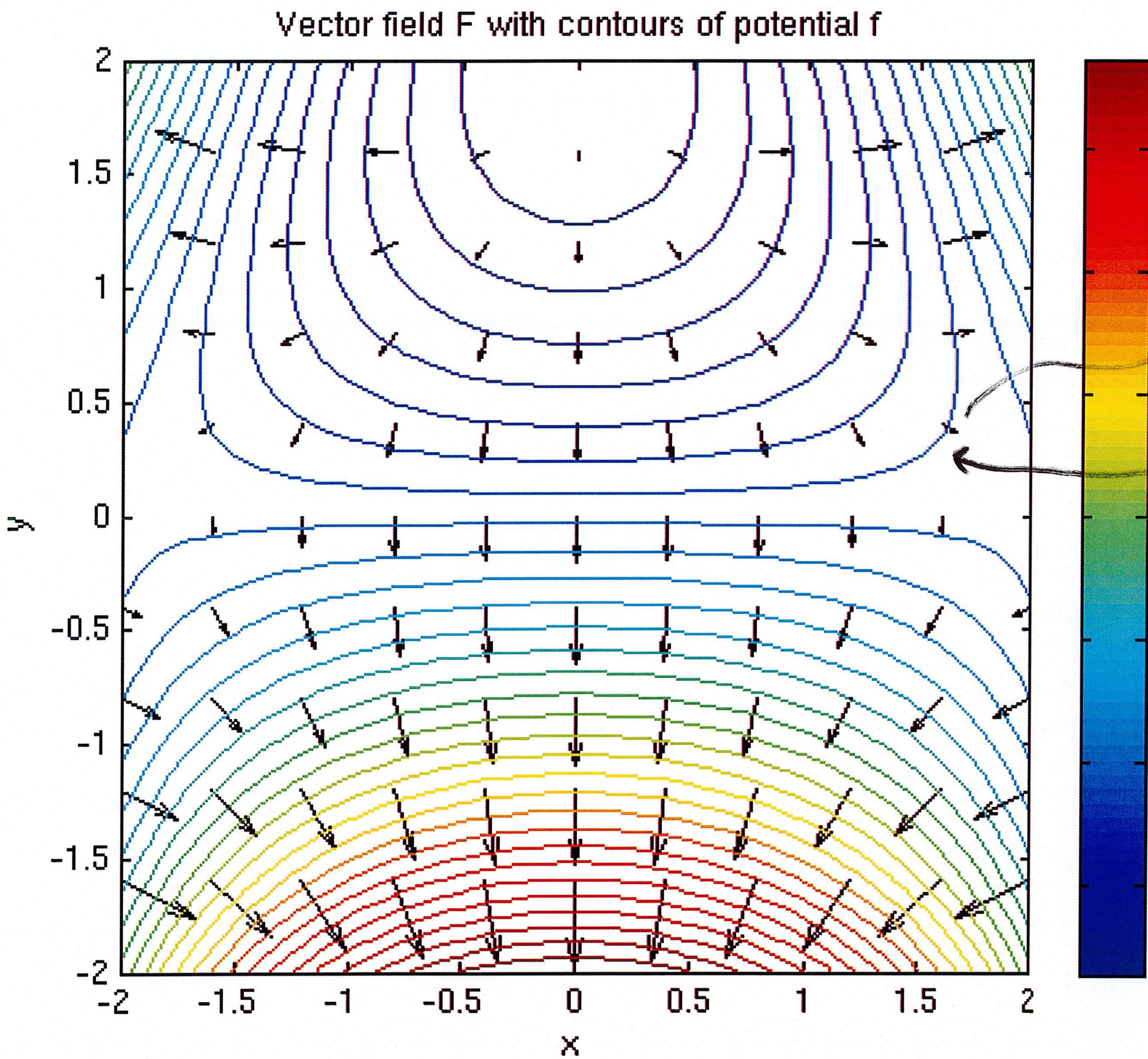
<u>example</u> $\vec{F} = \left\langle \frac{-x}{(\sqrt{x^2+y^2})^3}, \frac{-y}{(\sqrt{x^2+y^2})^3} \right\rangle$

notice it's the gradient of $U = \frac{1}{\sqrt{x^2+y^2}}$

$$\vec{\nabla} U = \left\langle \underbrace{-\tfrac{1}{2}(x^2+y^2)^{-3/2}(2x)}_{\frac{\partial U}{\partial x}}, \underbrace{-\tfrac{1}{2}(x^2+y^2)^{-3/2}(2y)}_{\frac{\partial U}{\partial y}} \right\rangle$$

$$= \left\langle \frac{-x}{(\sqrt{x^2+y^2})^3}, \frac{-y}{(\sqrt{x^2+y^2})^3} \right\rangle$$

so, we call the scalar field U the <u>potential function</u> of the vector field $\vec{F}$



vectors from the vector field

~~lele~~ level curve of the potential function

remember gradient is perpendicular to level curve $f(x,y) = C$

NOT every vector field has a potential function

because not every vector field is the gradient of something

if the vector field $\vec{F}$ represents a force and it has a potential function U $(\vec{F} = \vec{\nabla} U)$, then the force is said to be <u>conservative</u>

(one consequence: work done by force is independent of path)

gravity is one such conservative force

the work done by/against gravity is path-independent