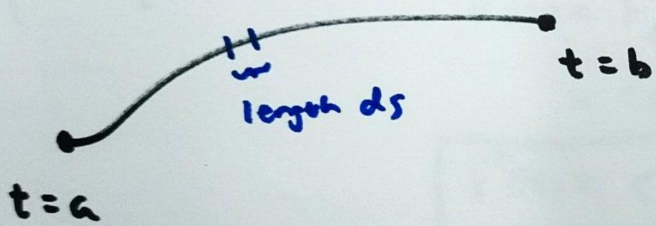
17.2 (part 1) Line Integrals of Functions

remember the length of $\vec{r}(t)$ $\quad a \le t \le b$ is

$$\int_a^b \underbrace{|\vec{r}\,'(t)|\,dt}_{\text{length of tiny segment of } \vec{r}(t)} = \int_C ds$$

C: parametrization of $\vec{r}(t)$



the length is a special case of the **line integral**: $\boxed{\int_C f(x,y)\,ds}$

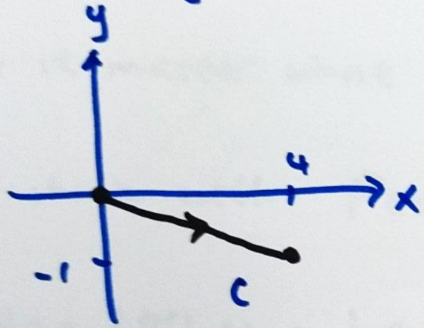
if $f(x,y)$ is the density of a wire with shape given by C,

then $\int_C f(x,y)\,ds$ is the mass of the wire

in 3D $\int_C f(x,y,z)\,ds$

finding C: the parametrization of the curve is crucial

<u>example</u> $\displaystyle\int_C x e^{y^2}\, ds$ C: line segment from $(0,0)$ to $(4,-1)$



parametrize C: find $\vec{r}(t)$, $a \le t \le b$ to describe the curve

C here is a line through $(0,0)$ to $(4,-1)$

$$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$$
$$= \langle 0, 0\rangle + t\langle 4, -1\rangle \qquad 0 \le t \le 1$$

$$\boxed{\vec{r}(t) = \langle \underset{x}{4t}, \underset{y}{-t}\rangle, \quad 0 \le t \le 1}$$

$$ds = |\vec{r}\,'|\,dt = |\langle 4, -1\rangle|\,dt = \sqrt{17}\,dt$$

y component of $\vec{r}(t)$

$$\int_C x e^{y^2}\, ds = \int_0^1 (4t)\, e^{(-t)^2} \sqrt{17}\, dt = \int_0^1 4\sqrt{17}\, t\, e^{t^2}\, dt$$

$\sqrt{17}\,dt$

x component of parametrization of C

$$= \cdots = \boxed{2\sqrt{17}\,(e-1)}$$

note $\vec{r}(t) = \langle 4t, -t \rangle$ $\quad 0 \le t \le 1$ is NOT the only way to describe the line segment

also correct: $\vec{r}(t) = \langle 8t, -2t \rangle$ $\quad 0 \le t \le \frac{1}{2}$

does it matter what parametrization we use in $\int_C f(x,y)\,ds$?

let's check with $\vec{r}(t) = \langle \underset{x}{8t}, \underset{y}{-2t} \rangle$ $\quad 0 \le t \le 1/2$ in $\int_C x e^{y^2}\,ds$

$$ds = |\vec{r}\,'|\,dt = |\langle 8, -2 \rangle|\,dt = \sqrt{68}\,dt \qquad 0 \le t \le 1/2$$

$$\int_C x e^{y^2}\,ds = \int_0^{1/2} (8t) e^{(-2t)^2} \sqrt{68}\,dt = 8\sqrt{68} \int_0^{1/2} t e^{4t^2}\,dt$$

$u = 4t^2$

$du = 8t\,dt$

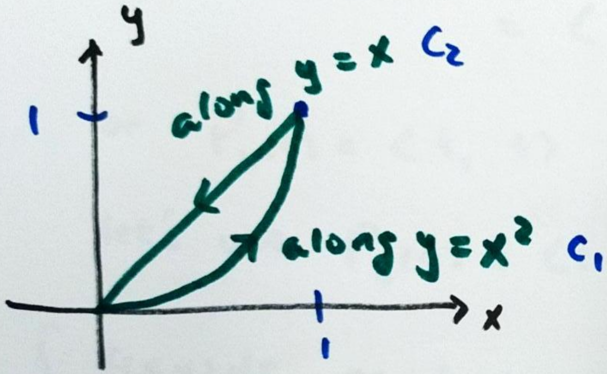
$$= 8\sqrt{68} \int_0^1 \frac{1}{8} e^u\,du$$

$$= 8\sqrt{68} \cdot \frac{1}{8} e^u \Big|_0^1 = \sqrt{68}\,(e-1)$$

$$= 2\sqrt{17}\,(e-1)$$ Same as before!

the parametrization of C does NOT change the answer __example__ $\int_C (x+\sqrt{y})\,ds$

C: from $(0,0)$ to $(1,1)$ along $y=x^2$
then $(1,1)$ to $(0,0)$ along $y=x$



need two parametrizations: ~~s~~ one for parabolic, one for line

C_1: $y=x^2 \rightarrow$ pick x to be something, for example t, then make y the square of that

$\vec{r}_1(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$

make sure you have the correct start, end, and direction

or $\vec{r}_1(t) = \langle \sqrt{t}, t \rangle \quad 0 \le t \le 1$

or $\vec{r}_1(t) = \langle 2t, 4t^2 \rangle \quad 0 \le t \le 1/2$

let's go with $\vec{r}_1(t) = \langle t, t^2 \rangle \quad 0 \le t \le 1$ for this example

C_2 : line segment from $(1,1)$ to $(0,0)$

$$\vec{r}_2(t) = \vec{r}_0 + t\vec{V} = \langle 1, 1\rangle + t\langle -1, -1\rangle \quad 0 \le t \le 1$$

$$= \langle 1-t, 1-t\rangle \quad 0 \le t \le 1$$

or $\vec{r}_2(t) = \langle t, t\rangle$ t : from 1 to 0

let's use $\vec{r}_2(t) = \langle 1-t, 1-t\rangle$ $0 \le t \le 1$ for this example

$\int_C f(x,y)\,ds$ on C_1 : $\vec{r}_1(t) = \langle \overset{x}{t}, \overset{y}{t^2}\rangle$ $0 \le t \le 1$

$$ds = |\vec{r}\,'|\,dt = |\langle 1, 2t\rangle|\,dt = \sqrt{1+4t^2}\,dt$$

on C_2 : $\vec{r}_2(t) = \langle \overset{x}{1-t}, \overset{y}{1-t}\rangle$ $0 \le t \le 1$

$$ds = |\vec{r}\,'|\,dt = \sqrt{2}\,dt$$

use x, y from $\vec{r}_1(t)$

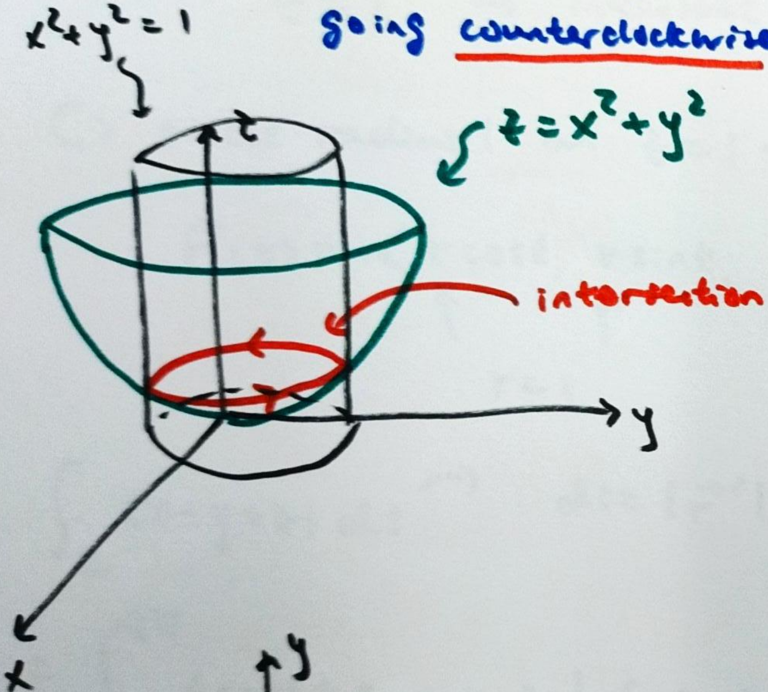
$$\int_C (x+\sqrt{y})\,ds = \underbrace{\int_0^1 \overbrace{(t+\sqrt{t^2})}^{\vec{r}_1(t)}\overbrace{\sqrt{1+4t^2}\,dt}^{ds}}_{C_1} + \underbrace{\int_0^1 (1-t+\sqrt{1-t}\,)\sqrt{2}\,dt}_{C_2}$$

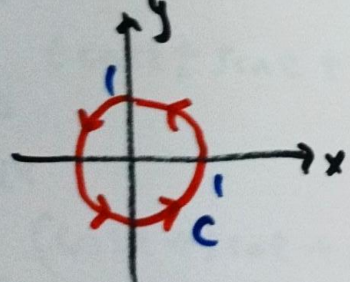
$$= \dots = \boxed{\frac{1}{6}(5\sqrt{5}-1) + \frac{7}{3\sqrt{2}}}$$

<u>example</u> $\int_C (x+y+z)\,ds$

C: intersection of paraboloid $z=x^2+y^2$ and the cylinder $x^2+y^2=1$

going <u>counterclockwise</u> when viewed from above





C is a circle because both surfaces have circular cross sections

How big is the circle C ?

$x^2+y^2=1$

$z=x^2+y^2$

the intersection is on the surface of $x^2+y^2=1$ so it's a circle radius 1 at what height is the intersection?

$x^2+y^2=1$

$z=x^2+y^2$

$\to$ $z=1$ $\to$ intersect at $z=1$

C: circle radius 1 on $z=1$

$$\vec{r}(t) = \langle r\cos t,\ r\sin t,\ 1\rangle = \langle \overset{x}{\cos t},\ \overset{y}{\sin t},\ \overset{z}{1}\rangle \quad 0 \le t \le 2\pi$$

$r=1$

or $\langle \cos(2t), \sin(2t), 1\rangle$ $\quad 0 \le t \le \pi$

$$\int_C (x+y+z)\,ds \qquad ds = |\vec{r}\,'|\,dt$$

$$= \int_0^{2\pi} (\cos t + \sin t + 1)\,|\langle -\sin t, \cos t, 0\rangle|\,dt$$

$$= \int_0^{2\pi} (\cos t + \sin t + 1)\,dt = \cdots = \boxed{2\pi}$$