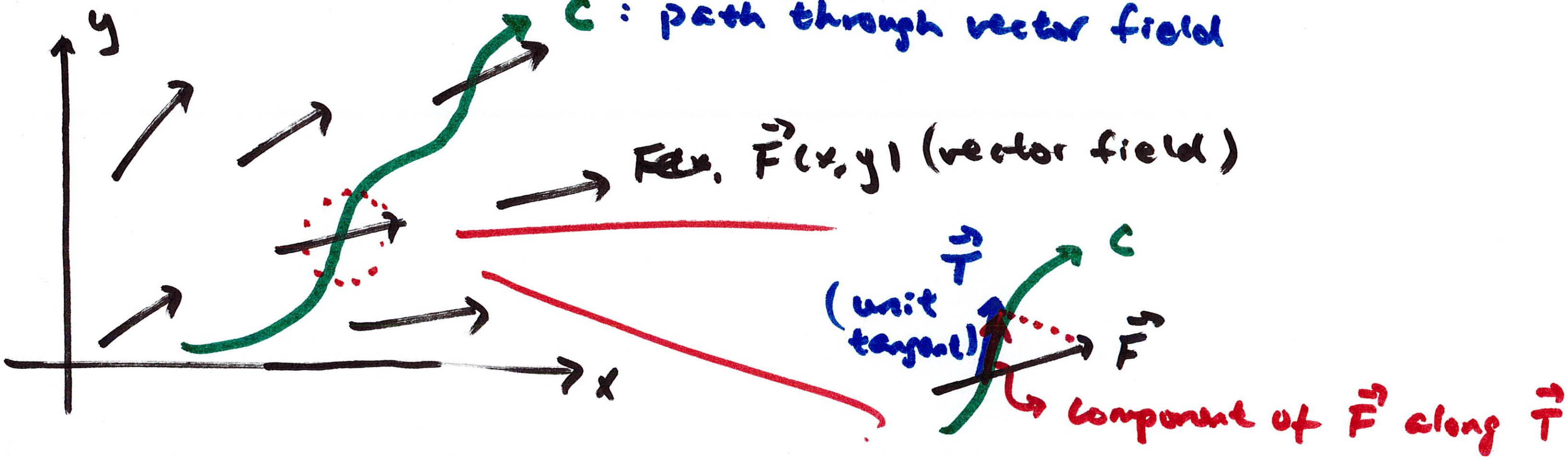
17.2 (part 2) Line Integrals in a Vector Field

last time: line integral in scalar field $\int_C f\,ds$

In a vector field, it's similar in many ways



we want to accumulate some of component of $\vec{F}$ along the path

most typically: vector field along the unit tangent of the path

→ accumulate $\vec{F}\cdot\vec{T}$

vector field

some small part of path

line integral: $\int_C \vec{F}\cdot\vec{T}\,ds$

unit tangent vector

How to calculate $\boxed{\int_C \vec{F}\cdot\vec{T}\,ds\ ?}$

parametrize C: $\vec{r}(t)$, $a \le t \le b$

then the unit tangent is $\vec{T} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$ and we know $ds = |\vec{r}\,'|dt$

$\int_C \vec{F}\cdot\vec{T}\,ds$ becomes $\int_C \vec{F}\cdot\dfrac{\vec{r}\,'}{|\vec{r}\,'|}|\vec{r}\,'|dt = \boxed{\int_a^b \vec{F}\cdot\vec{r}\,'dt}$

$$\vec{r}\,'dt = \frac{d\vec{r}}{dt}dt = d\vec{r}$$

another equivalent form: $\boxed{\int_C \vec{F}\cdot d\vec{r}}$

all boxed expressions are the same integral

common application: work done by the force vector field $\vec{F}$
in moving some object along C

example $\vec{F} = \langle xy, y-x \rangle$

C : line segment from $(0,1)$ to $(2,4)$

first step: parametrize C

here, line segment $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$= \langle 0, 1 \rangle + t \langle 2, 3 \rangle$

$\vec{r}(t) = \langle 2t, 1+3t \rangle \quad 0 \leq t \leq 1$

remember, the parametrization is NOT unique

(here, $\vec{r}(t) = \langle 4t, 1+6t \rangle \quad 0 \leq t \leq 1/2$ $
is also ok)

for this example, let's use $\vec{r}(t) = \langle 2t, 1+3t \rangle \quad 0 \leq t \leq 1$

line integral

$$\int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_C \vec{F} \cdot d\vec{r}$$

the middle expression looks easiest to use

$\vec{r}\,' = \langle 2, 3 \rangle$

$$\vec{F} = \langle xy,\ y-x \rangle = \Big\langle (2t)(1+3t),\ (1+3t)-(2t) \Big\rangle = \langle 6t^2+2t,\ t+1 \rangle$$

↗ ↑ ↗ ↗

use x, y of parametrization $\vec{r}(t)$

$$\int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_0^1 \langle 6t^2+2t,\ t+1\rangle \cdot \langle 2,\ 3\rangle\,dt$$

$$= \int_0^1 (12t^2 + 4t + 3t + 3)\,dt = \cdots = \boxed{\frac{25}{2}}$$

if C is a closed loop (start and end are same point)

then the line integral $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'dt = \int_C \vec{F}\cdot d\vec{r}$

is also called the <u>circulation</u> of $\vec{F}$ on C.

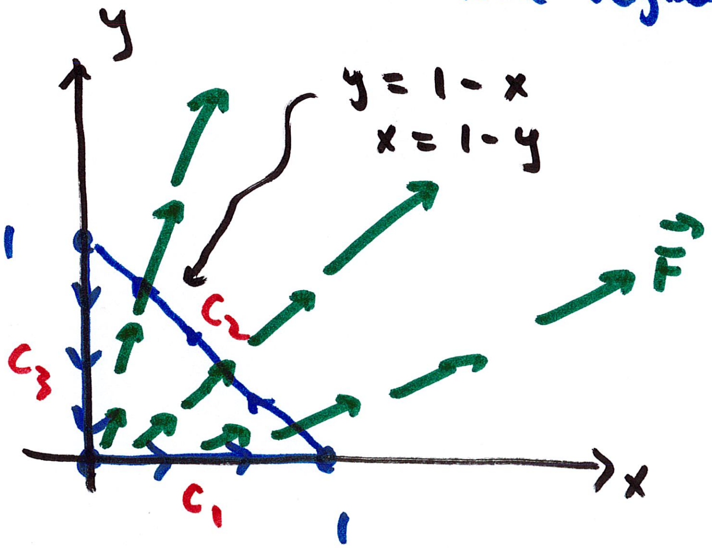
the calculation is basically the same

<u>example</u> $\vec{F} = \langle x, y\rangle$

C : line segment from (0, 0) to (1, 0) C_1

then line segment from (1, 0) to (0, 1) C_2

then line segment from (0, 1) ~~back to~~ back to (0, 0) C_3



C_1 : $\vec{r}(t) = \langle t, 0\rangle \quad 0 \le t \le 1$

C_2 : $\vec{r}(t) = \langle 1-t, t\rangle \quad 0 \le t \le 1$

C_3 : $\vec{r}(t) = \langle 0, 1-t\rangle \quad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \underline{\int_C \vec{F}\cdot\vec{r}\,'\,dt} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \underbrace{\langle t, 0\rangle}_{\vec{F} \text{ w/ } x,y \text{ of } C_1}\cdot\underbrace{\langle 1, 0\rangle}_{\vec{r}\,' \text{ on } C_1}\,dt + \underbrace{\int_0^1 \langle 1-t, t\rangle\cdot\langle -1, 1\rangle\,dt}_{C_2} + \underbrace{\int_0^1 \langle 0, 1-t\rangle\cdot\langle 0, -1\rangle\,dt}_{C_3}$$

$$= \int_0^1 t\,dt + \int_0^1 (2t-1)\,dt + \int_0^1 (t-1)\,dt = \cdots = \boxed{0}$$

if $\vec{F} = \langle f, g \rangle$

and $\vec{r}(t) = \langle x(t), y(t) \rangle$ $\quad a \le t \le b$

then $\vec{T} = \dfrac{\vec{F}'}{|\vec{r}'|} = \dfrac{\langle x', y' \rangle}{\sqrt{(x')^2 + (y')^2}}$ $\qquad ds = |\vec{r}'|\,dt = \sqrt{(x')^2 + (y')^2}\,dt$

Sub into $\displaystyle\int_C \vec{F} \cdot \vec{T}\,ds$

$$= \int_C \langle f, g \rangle \cdot \frac{\langle x', y' \rangle}{\cancel{\sqrt{(x')^2 + (y')^2}}} \cancel{\sqrt{(x')^2 + (y')^2}}\,dt$$

$$= \int_C \langle f, g \rangle \cdot \langle x', y' \rangle\,dt = \int_C (f \overset{\frac{dx}{dt}}{x'} + g \overset{\frac{dy}{dt}}{y'})\,dt$$

$$= \int_C \left(f \frac{dx}{dt} + g \frac{dy}{dt}\right) dt = \boxed{\int_C f\,dx + g\,dy}$$

equivalent expression of $\displaystyle\int_C \vec{F} \cdot \vec{T}\,ds$

$\int_C f\,dx + g\,dy$ tells us the vector field is $\vec{F} = \langle f, g \rangle$

and path $\vec{r}(t) = \langle x, y \rangle$

for example, $\int_C \overset{f}{\boxed{xy}}\,dx + \overset{g}{\boxed{(x+y)}}\,dy$ $\quad \vec{F} = \langle f, g \rangle = \langle xy, x+y \rangle$

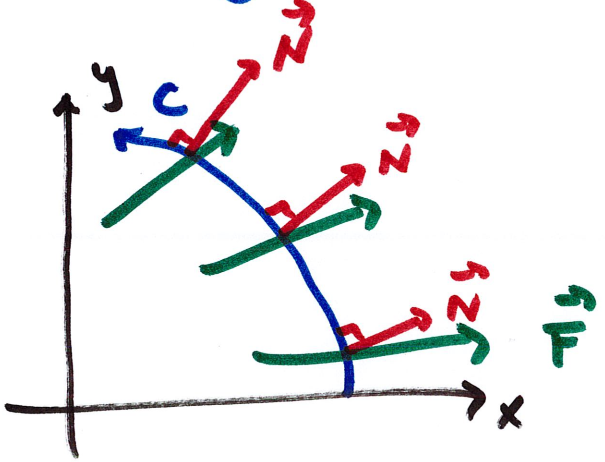
C : (0, 0) to (1, 1) along $y = x^2$

C : $\vec{r}(t) = \langle \underset{\overset{\swarrow}{x(t)}}{t}, \underset{\overset{\downarrow}{y(t)}}{t^2} \rangle \qquad 0 \le t \le 1$

then calculate as in previous examples

if we replace $\vec{T}$ with the unit normal $\vec{N}$ (instead of the unit tangent)

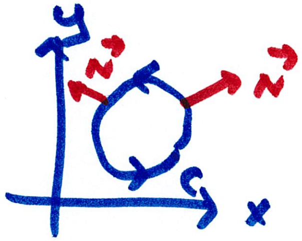
then $\int_C \vec{F}\cdot\vec{T}\,ds$ becomes $\int_C \vec{F}\cdot\vec{N}\,ds$ → "flux integral"

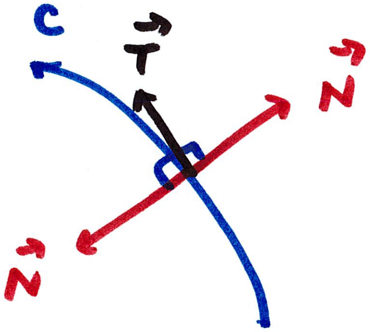


$\int_C \vec{F}\cdot\vec{N}\,ds$ accumulates the portion of $\vec{F}$ *through* the path

(think of it as air/water flowing through a barrier in the shape of C)

there are *two* unit normals on $\vec{C}$





which one to use?

convention: if C is closed loop, choose $\vec{N}$ that points *outward*

if C is not closed, choose $\vec{N}$ that is to the *right* of $\vec{T}$