17.3 Conservative Vector Fields and the Fundamental Theorem of Line Integrals

if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi$ $\qquad$ ϕ: potential function

given ϕ finding $\vec{F}$ is easy

harder: given $\vec{F}$, how do we know if it is conservative?

and if so, how to find ϕ?

let $\vec{F} = \langle f, g \rangle$ be a conservative vector field

then we know $\vec{F} = \langle f, g \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$

this means $f = \frac{\partial \phi}{\partial x}$ and $g = \frac{\partial \phi}{\partial y}$

furthermore, we know that the mixed partial derivatives are equal

$$\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial \phi}{\partial x}}_{f}\Big) = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial \phi}{\partial y}}_{g}\Big)$$

so, if $\vec{F} = \langle f, g \rangle$ <u>is</u> conservative, then $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ or $f_y = g_x$

example $\vec{F} = \langle x, y \rangle$

is this conservative?

$$\left.\begin{aligned} f &= x \\ g &= y \end{aligned}\right\} \text{ check if } f_y = g_x \text{ ?}$$

$f_y = 0$ $\quad g_x = 0$ $\quad$ yes, therefore $\vec{F} = \langle x, y \rangle$ is conservative

(so there is ϕ such that

$\vec{\nabla}\phi = \langle x, y \rangle$)

example $\vec{F} = \langle -y, x \rangle$

$$\left.\begin{aligned} f &= -y \\ g &= x \end{aligned}\right\} \text{ is } f_y = g_x \text{ ?}$$

$f_y = -1$ $\quad g_x = 1$

$f_y \neq g_x$ so $\vec{F} = \langle -y, x \rangle$ is <u>NOT</u> conservative

there is no ϕ such that $\vec{\nabla}\phi = \vec{F}$

how to find ϕ if we know $\vec{F}$ is conservative?

example $\vec{F} = \langle \underbrace{x+y}_{f}, \underbrace{x}_{g} \rangle$

is $f_y = g_x$? $f_y = 1$ $g_x = 1$ yes, so there is ϕ such $\vec{\nabla}\phi = \langle x+y, x \rangle$

now let's find ϕ

$$\vec{F} = \langle x+y, x \rangle = \vec{\nabla}\phi = \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rangle$$

$$x+y = \frac{\partial \phi}{\partial x} \quad (1)$$

$$x = \frac{\partial \phi}{\partial y} \quad (2)$$

from (1), integrate with respect to x (y is constant)

$$\phi = \int (x+y)\, dx = \frac{1}{2}x^2 + xy + \underbrace{a(y)}_{} \quad (3)$$

y is constant

function that contains y because this would be treated as a constant in $\frac{\partial \phi}{\partial x}$

③ is ϕ but we need to know what $a(y)$ is

its partial with y must be equal to ②: $\underbrace{\dfrac{\partial \phi}{\partial y} = x}$

③: $\phi = \frac{1}{2}x^2 + xy + a(y)$

$$\frac{\partial \phi}{\partial y} = x + \frac{da}{dy} = \underbrace{x}_{\text{from ②}}$$

so $\dfrac{da}{dy} = 0 \rightarrow a = C$ (a true constant)

so, from ③ again, we get $\boxed{\phi = \frac{1}{2}x^2 + xy + C}$

check: is $\vec{\nabla}\phi = \vec{F} = \langle x+y, x \rangle$?

$\vec{\nabla}\phi = \langle x+y, x \rangle$ yes so our ϕ is correct.

3D : $\vec{F} = \langle f, g, h \rangle$

how to check if $\vec{F}$ is conservative?

if $\vec{F}$ is conservative, then $\vec{\nabla}\phi = \vec{F}$

$$\vec{F} = \langle f, g, h \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

$f = \phi_x$

$g = \phi_y$

$h = \phi_z$

mixed partials are equal :

$$(\overset{f}{\phi_x})_y = (\overset{g}{\phi_y})_x$$

$$(\overset{g}{\phi_y})_z = (\overset{h}{\phi_z})_y$$

$$(\overset{h}{\phi_z})_x = (\overset{f}{\phi_x})_z$$

if these 3 pairs are equal, then $\vec{F}$ is conservative

$\vec{F} = \langle f, g, h \rangle$ is conservative if

$$f_y = g_x$$

$$g_z = h_y$$

$$h_x = f_z$$

example $\vec{F} = \langle \underbrace{x^2 - ze^y}_{f}, \underbrace{y^3 - xze^y}_{g}, \underbrace{z^4 - xe^y}_{h} \rangle$

is $f_y = g_x$? $\quad -ze^y = -ze^y$ yes

$g_z = h_y$? $\quad -xe^y = -xe^y$ yes

$f_z = h_x$? $\quad -e^y = -e^y$ yes

ALL 3 must be yes for $\vec{F} = \nabla \phi$

now we know

$\vec{F} = \langle f, g, h \rangle = \langle \phi_x, \phi_y, \phi_z \rangle$

$\phi_x = x^2 - ze^y$ ①

$\phi_y = y^3 - xze^y$ ②

$\phi_z = z^4 - xe^y$ ③

from ①: $\phi = \int (x^2 - ze^y)\, \underbrace{dx}_{\text{y, z are constants}} = \frac{1}{3}x^3 - xze^y + \underbrace{a(y,z)}_{\text{function that can depend on y, z}}$ ④

what is a?

partial of ④ with y must be ②

$\phi_y = -xze^y + \frac{\partial a}{\partial y} = \underbrace{y^3 - xze^y}_{②} \rightarrow \frac{\partial a}{\partial y} = y^3$ ⑤

partial of ④ with z must be ③

$\phi_z = -xe^y + \frac{\partial a}{\partial z} = \underbrace{z^4 - xe^y}_{③} \rightarrow \frac{\partial a}{\partial z} = z^4$ ⑥

integrate ⑤ with y : $a = \frac{1}{4}y^4 + b(z)$

take partial with z and compare to ⑥

$\frac{\partial a}{\partial z} = \frac{db}{dz} = z^4 \rightarrow b = \frac{1}{5}z^5 + C$ so $a = \frac{1}{4}y^4 + \frac{1}{5}z^5 + C$

so $\boxed{\phi = \frac{1}{3}x^3 - xze^y + \frac{1}{4}y^4 + \frac{1}{5}z^5 + C}$

why bother ~~wait~~ with ϕ ?

because if $\vec{F}$ is conservative, then the line integral

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$$ is <u>path independent</u>

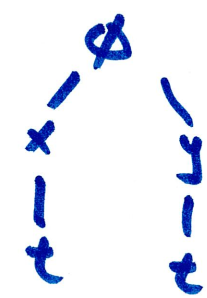
why? $\vec{F}\cdot\vec{T}\,ds = \vec{F}\cdot\vec{r}\,'\,dt$

$$= \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \underbrace{\left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} \right)}\,dt$$

$$= \frac{d\phi}{dt}\,dt = d\phi$$

Chain Rule



$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C d\phi = \phi(B) - \phi(A)$$

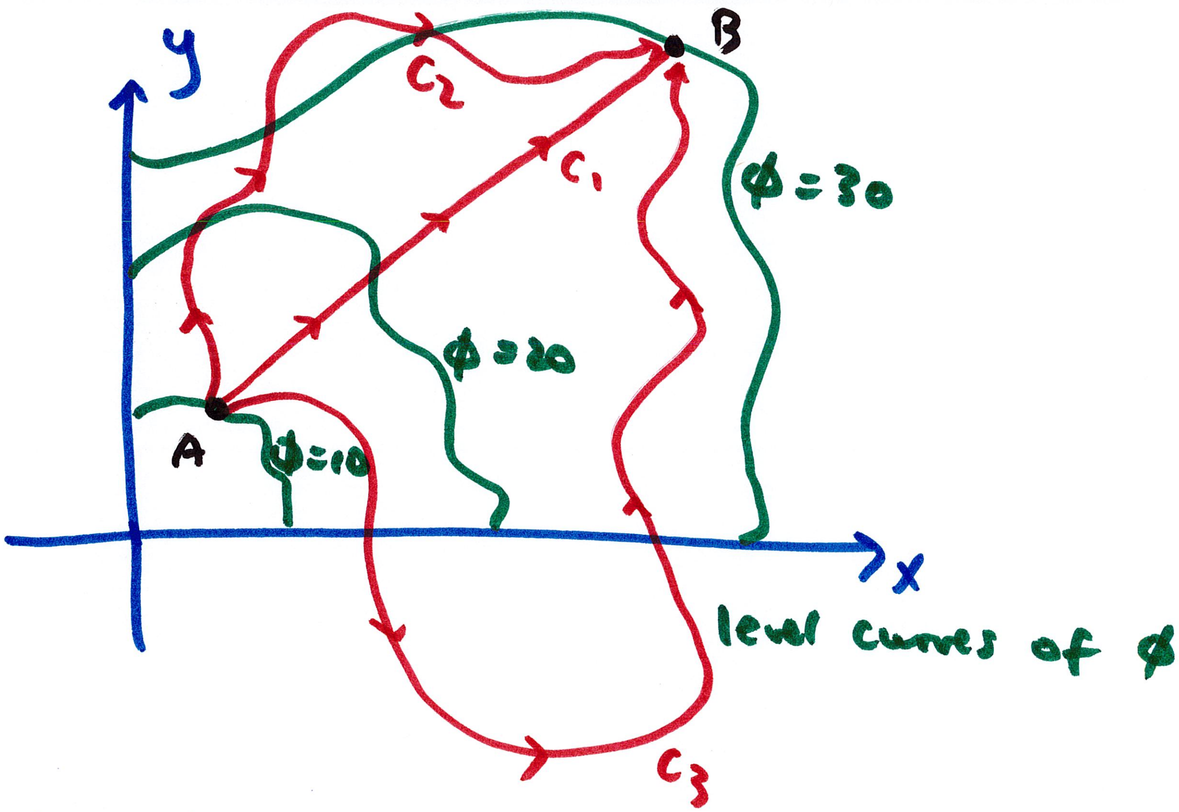
end location

start location

Fundamental Theorem of Line Integrals

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \vec{F} \cdot \vec{r}\,'\, dt = \int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

<u>if</u> $\vec{F} = \vec{\nabla} \phi$ (if $\vec{F}$ is conservative)



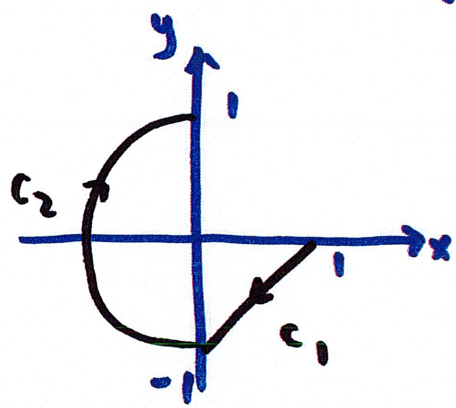
all paths C_1, C_2, C_3 produce the same value of $\int_C \vec{F} \cdot \vec{T}\, ds$ because the start (A) and end (B) are the same

$$\int_C \vec{F} \cdot \vec{T}\, ds = \phi(B) - \phi(A) = 30 - 10 = 20$$

<u>example</u> $\int_C \vec{F}\cdot d\vec{r}$ $\vec{F} = \langle x+y, x\rangle$

C: line segment from (1, 0) to (0, -1)

then along the left half of $x^2+y^2=1$ to (0, 1)



let's first try as a regular line integral

parametrize: $C_1: \vec{r}(t) = \langle 1-t, t\rangle \quad 0\le t\le 1$

$C_2: \vec{r}(t) = \langle -\sin t, -\cos t\rangle \quad 0\le t\le \pi$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}\,'dt$$

$$= \underbrace{\int_0^1 \langle 1-2t, 1-t\rangle\cdot\langle -1, -1\rangle\,dt}_{C_1} + \underbrace{\int_0^{\pi} \langle -\sin t-\cos t, -\sin t\rangle\cdot\langle -\cos t, \sin t\rangle\,dt}_{C_2}$$

$$= \int_0^1 (3t-2)\,dt + \int_0^{\pi}(\sin t\cos t + \cos^2 t - \sin^2 t)\,dt = \dots = \boxed{-\tfrac{1}{2}}$$

not terribly bad, but we can use the Fundamental Theorem of Line Integrals to get the answer MUCH quicker

$\vec{F} = \langle x+y, x \rangle$

we know this is conservative from the earlier example and

$\phi = \frac{1}{2}x^2 + xy + C$

so, $\int_C \vec{F} \cdot \vec{r}\,' dt = \phi(B) - \phi(A)$

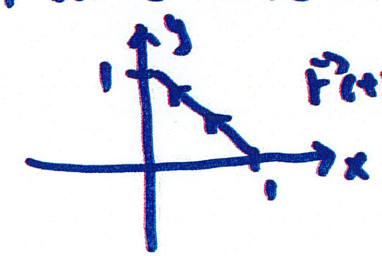
end location $(0,1)$ (x, y)

start location $(1,0)$ (x, y)

$$= \left[\frac{1}{2}(0)^2 + (0)(1) + C\right] - \left[\frac{1}{2}(1)^2 + (1)(0) + C\right] = \boxed{-\frac{1}{2}}$$

alternatively, since path doesn't matter, we could have chosen a simpler path w/ same start and end in the line integral

for example,



$\vec{r}(t) = \langle 1-t, t \rangle$