17.4 Green's Theorem

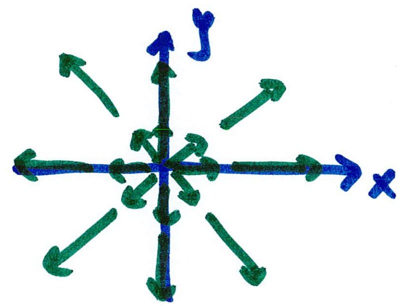
let $\vec{F} = \langle f, g \rangle$ be a 2D vector field

the quantity $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k} = \langle 0, 0, g_x - f_y \rangle$ is called the curl of $\vec{F}$

written as $\text{curl}\,\vec{F}$

$|\text{curl}\,\vec{F}| = g_x - f_y$ is a measure of rotation in the vector field

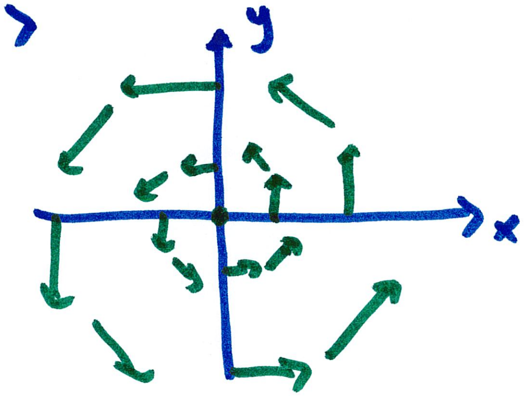
for example. $\vec{F} = \langle \underset{f}{x}, \underset{g}{y} \rangle$



$\text{curl}\,\vec{F} = \langle 0, 0, 0-0 \rangle = \vec{0}$ $\quad |\text{curl}\,\vec{F}| = 0$ indicates no rotation in $\vec{F}$

confirmed by visualization

$\vec{F} = \langle -y, x \rangle$
$\quad\quad\;\; f \quad\; g$



clearly a rotation present

$\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle$

$= \langle 0, 0, 1-(-1) \rangle$

$= \langle 0, 0, 2 \rangle$

$|\text{curl}\,\vec{F}| = 2 \neq 0$ so indicating rotation is present

recall if $g_x - f_y = 0$ then $\vec{F} = \langle f, g \rangle$ is conservative ($\vec{F} = \vec{\nabla}\phi$)

if $|g_x - f_y| = 0$, then $\text{curl}\,\vec{F} = \vec{0}$

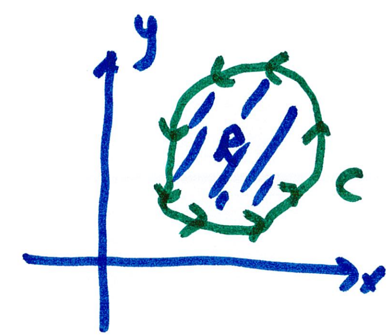
but $g_x - f_y = 0$ then $\vec{F}$ is conservative

so a conservative vector field is irrotational (no rotation, $|\text{curl}\,\vec{F}| = 0$)

Green's Theorem

if $\vec{F} = \langle f, g \rangle$ and C is a simple closed path traversed once in the counter-clockwise direction, then

$$\oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$



indicates path C is closed

region enclosed by C

why is this true?

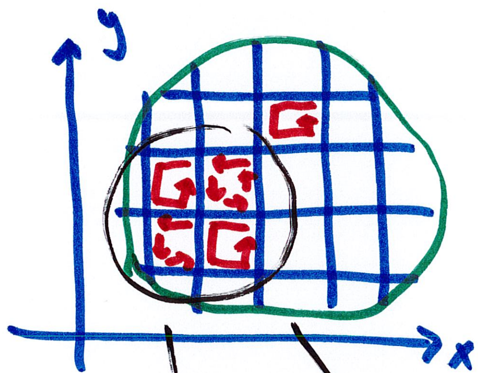
left side: $\oint_C f\,dx + g\,dy = \oint_C \vec{F} \cdot \vec{T}\,ds = \oint_C \vec{F} \cdot \vec{r}\,'\,dt = \oint_C \vec{F} \cdot d\vec{r}$

$\langle f, g \rangle$ $\qquad$ $\langle dx, dy \rangle$

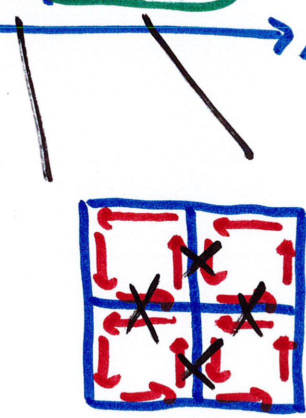
the left side accumulates the component of $\vec{F}$ along C on the closed loop C $\rightarrow$ circulation right side: $\iint_R \underbrace{\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)}_{|\mathrm{curl}\,\vec{F}|} dA$

$|\mathrm{curl}\,\vec{F}|$ (measures rotation)

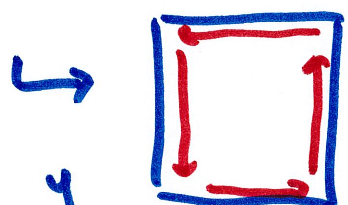
this integral accumulates rotation in R



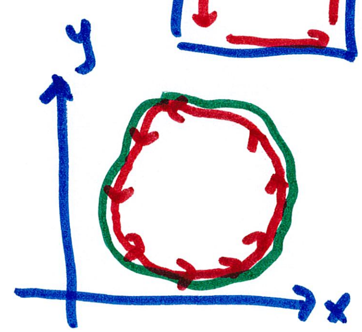
we accumulate the curl (red things) from grid to grid



neighboring flows cancel out



equivalent to

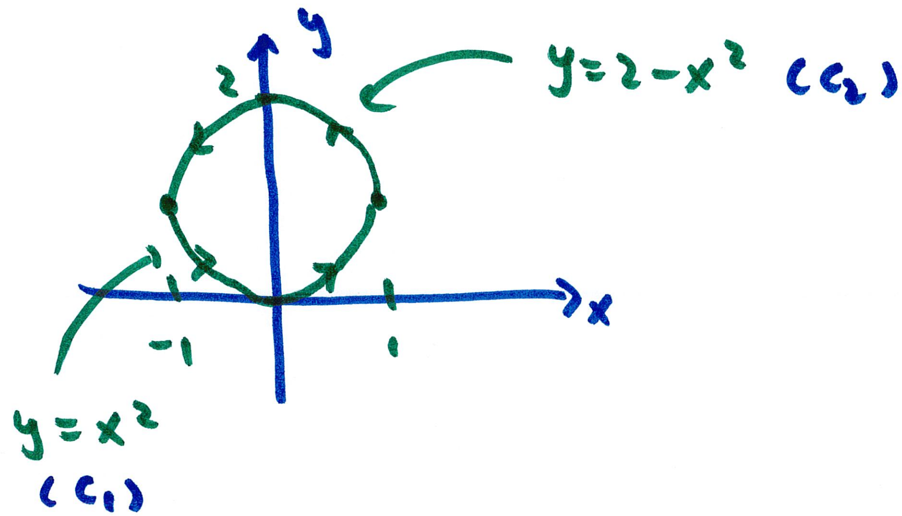


this is what $\oint_C f\,dx + g\,dy$ does.

This shows why Green's Theorem is true

example $\vec{F} = \langle y+2,\ x^2+1 \rangle$

C : $(-1, 1)$ to $(1, 1)$ along $y = x^2$ then back to $(-1, 1)$ along $y = 2 - x^2$



let's do this as a line integral first, then try Green's Theorem

C_1 : $\vec{r} = \langle t, t^2 \rangle \quad -1 \le t \le 1$

C_2 : $\vec{r} = \langle -t, 2-t^2 \rangle \quad -1 \le t \le 1$

line integral:
$$\oint_C \vec{F}\cdot\vec{r}\,'\,dt = \overbrace{\int_{-1}^{1} \underbrace{\langle t^2+2,\ t^2+1 \rangle}_{\substack{\vec{F} \text{ w/ } x,\,y \\ \text{from } \vec{r} \text{ on } C_1}} \cdot \langle 1, 2t \rangle\,dt}^{C_1}$$

$$+ \underbrace{\int_{-1}^{1} \langle 4-t^2,\ t^2+1 \rangle \cdot \langle -1, -2t \rangle\,dt}_{C_2}$$

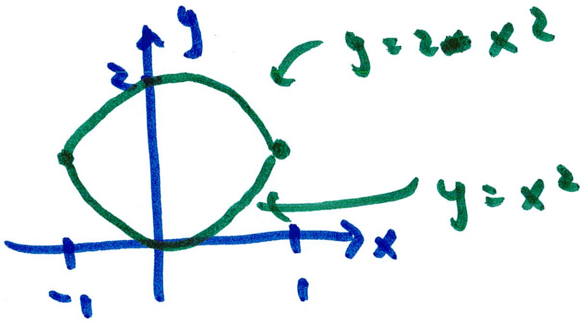
$$= \int_{-1}^{1} (2t^3 + t^2 + 2t + 2)\,dt + \int_{-1}^{1} (-2t^3 + t^2 - 2t - 4)\,dt$$

$$= \cdots = \frac{14}{3} - \frac{22}{3} = \boxed{-\frac{8}{3}}$$

now let's compare w/ using Green's Theorem $\oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$

$\vec{F} = \langle \underbrace{y+2}_{f}, \underbrace{x^2+1}_{g} \rangle$ $\quad g_x - f_y = 2x - 1$

R:



$-1 \le x \le 1$

$x^2 \le y \le 2 - x^2$

$$\iint_R (g_x - f_y)\,dA = \int_{-1}^{1}\int_{x^2}^{2-x^2} (2x-1)\,dy\,dx = \int_{-1}^{1} (2x-1)(2-2x^2)\,dx$$

$$= \int_{-1}^{1} (-4x^3 + 2x^2 + 4x - 2)\,dx = \cdots = \boxed{-\frac{8}{3}}$$

Green's Theorem allows us to calculate the area of R using a line integral

$$\oint_C f\,dx + g\,dy = \iint_R \underbrace{(g_x - f_y)}\,dA$$

if $g_x - f_y = 1$

right side is $\iint_R dA$ = area of R

so, come up with a $\vec{F} = \langle f, g \rangle$ such that $g_x - f_y = 1$

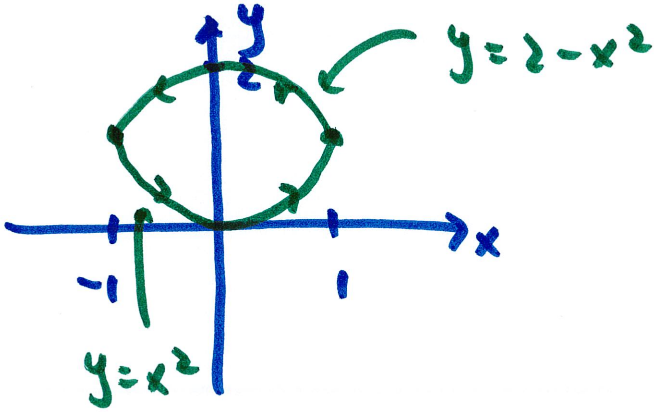
then $\oint_C f\,dx + g\,dy$ = area of R

many possibilities: for example, $\vec{F} = \langle \underset{f}{-\frac{1}{2}y}, \underset{g}{\frac{1}{2}x} \rangle$

using that $\vec{F}$, $\oint_C f\,dx + g\,dy = \boxed{\frac{1}{2}\oint_C x\,dy - y\,dx}$

this gives us the area of R as a line integral

example Area enclosed by C from previous example



same parametrization as before

$$C_1 : \vec{r} = \langle \overset{x}{t}, \overset{y}{t^2} \rangle \quad -1 \le t \le 1$$

$$C_2 : \vec{r} = \langle \underset{x}{-t}, \underset{y}{2-t^2} \rangle \quad -1 \le t \le 1$$

$$\text{area} = \frac{1}{2} \oint_C x\,dy - y\,dx$$

$$= \underbrace{\frac{1}{2} \int_{-1}^{1} \underset{x}{(t)} \underbrace{(2t)dt}_{dy} - \underset{y}{(t^2)} \underbrace{(dt)}_{dx}}_{C_1} + \underbrace{\frac{1}{2} \int_{-1}^{1} \underset{x}{(-t)} \underbrace{(-2t\,dt)}_{dy} - \underset{y}{(2-t^2)} \underbrace{(-dt)}_{dx}}_{C_2}$$

$$= \frac{1}{2} \int_{-1}^{1} t^2\,dt + \frac{1}{2} \int_{-1}^{1} (t^2+2)\,dt = \frac{1}{3} + \frac{7}{3} = \boxed{\frac{8}{3}}$$

if we replace $\vec{T}$ with $\vec{N}$ (unit normal) in Green's Theorem

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (g_x - f_y)\,dA$$

we end up with

$$\oint_C f\,dy - g\,dx = \iint_R (f_x + g_y)\,dA$$

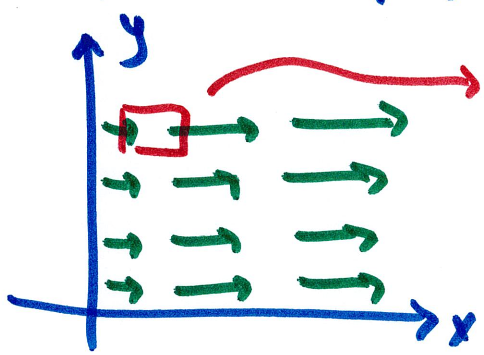
this allows us to calculate the flux integral as a double integral

$f_x + g_y$ is called the Divergence of $\vec{F} = \langle f, g \rangle$

written as $\operatorname{div}\vec{F}$

it measures the change of a small volume as it flows in the vector field

for example, $\vec{F} = \langle x, 0 \rangle \rightarrow \operatorname{div}\vec{F} = 1 \rightarrow$ means volume increases



right edge is tugged to the right more strongly than left (longer vector)