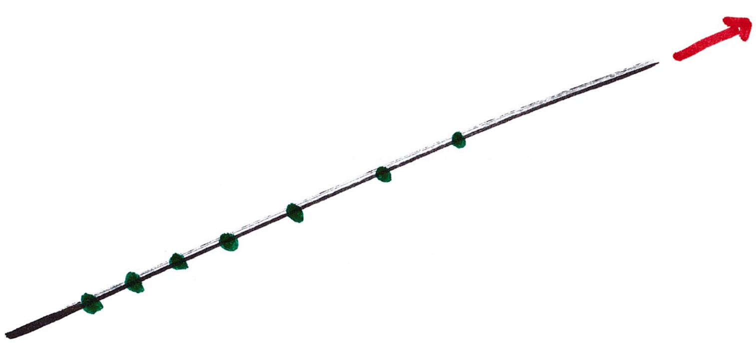
13.5 Lines and Planes in space

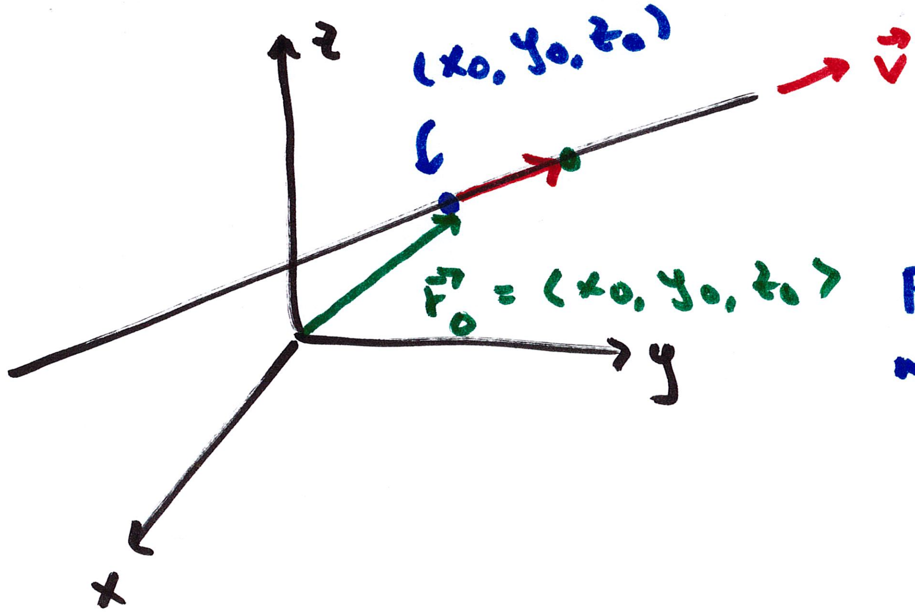
line: a collection of points that lie along a certain direction



$\vec{v} \to$ direction vector

how to write eq. of line?

use vectors to describe location of any point



position vector of one point

move along the direction vector to find next point

So, the <u>vector form</u> of equation of a line is

$$\boxed{\underbrace{\vec{r}(t) = \vec{r}_0}_{\text{position vector to one point on the line}} + \underbrace{t\vec{v}}_{\text{a step along the line in the direction of } \vec{v}}} \qquad -\infty < t < \infty$$

the tip of the vector $\vec{r}(t)$, as we change t, traces out all points on the line.

if $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$

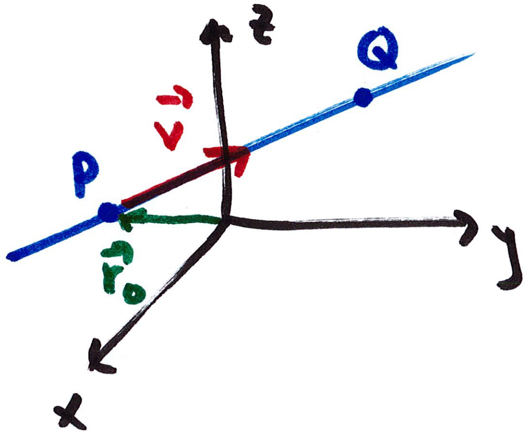
$\vec{v} = \langle a, b, c \rangle$

then $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$

$$= \langle \underbrace{x_0 + at}_{x}, \underbrace{y_0 + bt}_{y}, \underbrace{z_0 + ct}_{z} \rangle$$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{parametric form of line}$$

example Line through the points $P(0,1,2)$ and $Q(-3,4,7)$



position vector to one point

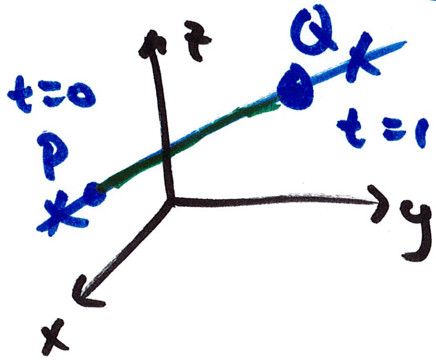
$\vec{r}_0 = \langle 0, 1, 2 \rangle$ Q is ok, too

direction vector : $\vec{v} = \vec{PQ} = \langle -3, 3, 5 \rangle$

$$\boxed{\begin{gathered}\vec{r}(t) = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle \\ -\infty < t < \infty\end{gathered}}$$ vector form

$$\left.\begin{aligned} x &= -3t \\ y &= 1+3t \\ z &= 2+5t \end{aligned}\right\}$$ parametric form. $-\infty < t < \infty$

what if we only want the line segment between P and Q?



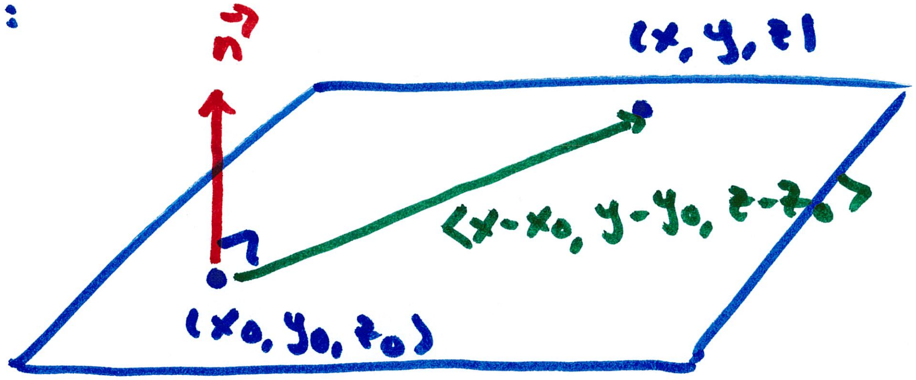
start w/ $\vec{r}(t) = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle$

note if $t=0$, $\vec{r}(0) = \langle 0, 1, 2 \rangle$ P

if $t=1$, $\vec{r}(1) = \langle -3, 4, 7 \rangle$ Q

so, $\boxed{\vec{r}(t) = \langle 0, 1, 2 \rangle + t \langle -3, 3, 5 \rangle \quad 0 \le t \le 1}$

Plane :



(x_0, y_0, z_0) one point in the plane

$\vec{n}$: vector perpendicular to the plane (normal vector)

(x, y, z) some other point in the plane

$\langle x-x_0, y-y_0, z-z_0 \rangle$ is a vector in the plane

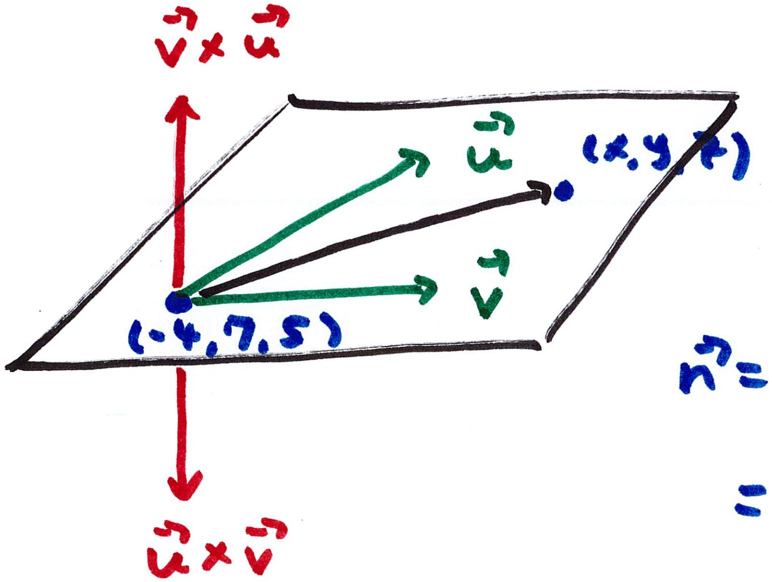
since $\vec{n} = \langle a, b, c \rangle$ is perpendicular to the plane, it must be perpendicular to $\langle x-x_0, y-y_0, z-z_0 \rangle$

so, $\langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

this is equation of a plane containing (x_0, y_0, z_0) and has normal vector $\langle a, b, c \rangle$

__example__ Equation of the plane containing the vectors $\vec{u} = \langle 0, 1, 2 \rangle$, $\vec{v} = \langle -1, -3, 0 \rangle$ and passes through the point $(-4, 7, 5)$



$\vec{n} = \vec{v} \times \vec{u}$ or $\vec{u} \times \vec{v}$

$$\vec{n} = \vec{v} \times \vec{u} = \langle -1, -3, 0 \rangle \times \langle 0, 1, 2 \rangle$$
$$= \langle -6, 2, -1 \rangle$$

So, eq. of this plane is

$\cancel{\langle -6, 2, -1 \rangle \cdot \langle}$

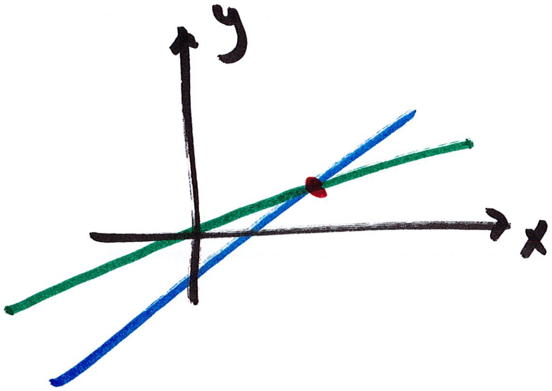
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

normal vector

$$-6(x - (-4)) + 2(y - 7) - 1(z - 5) = 0 \rightarrow \boxed{-6x + 2y - z = 33}$$

a bit more about lines

in $\mathbb{R}^2$, if two lines are not parallel, they eventually intersect at one point

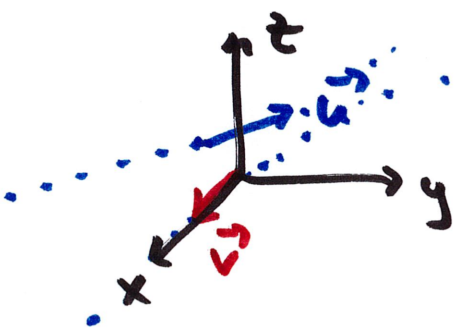


in $\mathbb{R}^3$, two lines are parallel if their direction vectors are parallel

if the direction vectors are not parallel, they <u>don't</u> necessarily intersect

for example, $\vec{u} = \langle 1, 0, 1 \rangle + t \langle 1, 2, 3 \rangle$

$\vec{v} = t \langle 1, 0, 0 \rangle$



different heights, never intersect

in this case, we say the lines are <u>skew</u>

Supplemental example

example Two objects travel on the lines

$$\vec{r}_1(t) = \langle 2t+3,\ 4t+2,\ 3t+5 \rangle \quad -\infty < t < \infty$$

$$\vec{r}_2(s) = \langle s+2,\ 3s-1,\ -5s+10 \rangle \quad -\infty < s < \infty$$

will the objects' paths intersect?

will the objects collide with each other?

intersect: can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$?

collide: can we find t, s such that $\vec{r}_1(t) = \vec{r}_2(s)$ AND $t = s$?

if $\vec{r}_1(t) = \vec{r}_2(s)$ then

x: $2t+3 = s+2$ — ①

y: $4t+2 = 3s-1$ — ②

z: $3t+5 = -5s+10$ — ③

from ①, $s = 2t+1$

Sub into ② $4t+2 = 3(2t+1)-1 \rightarrow \boxed{t = 0 \quad \text{so,} \quad s = 1}$ Check if these work in ③

③: $3(0)+5 = -5(1)+10$?

yes, so at $t=0$, $s=1$, $\vec{r}_1(t) = \vec{r}_2(s)$ they intersect