17.5 Curl and Divergence

gradient: $\underbrace{\vec{\nabla}}_{\text{must be a vector-like thing}} \underbrace{f}_{\text{scalar}} = \langle \underbrace{f_x, f_y, f_z}_{\text{vector}} \rangle$

$\vec{\nabla}$ is called the "del operator" and can be defined as

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

in 2D, $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$

$\vec{\nabla}$ is like a vector but means nothing until applied to something

so gradient is really

$$\vec{\nabla} f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Since $\vec{\nabla}$ behaves like a vector, we can cross and dot it with another vector

$$\boxed{\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}}$$

2D: $\vec{F} = \langle f, g, 0 \rangle$ last time: $\text{curl}\,\vec{F} = \langle 0, 0, g_x - f_y \rangle$

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle f, g, 0 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & 0 \end{vmatrix} = \left\langle \frac{\partial 0}{\partial y} - \frac{\partial g}{\partial z}, -\left(\frac{\partial 0}{\partial x} - \frac{\partial f}{\partial z}\right), \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right\rangle$$

zero because $\vec{F} = \langle f, g, 0 \rangle$ is 2D

$$= \langle 0, 0, g_x - f_y \rangle$$

agrees w/ how $\text{curl}\,\vec{F}$ was introduced last time try a 3D $\vec{F}$

<u>example</u> $\vec{F} = \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}(x^2y^3z) - \frac{\partial}{\partial z}(x^3yz^2),\ -\left(\frac{\partial}{\partial x}(x^2y^3z) - \frac{\partial}{\partial z}(xy^2z^3)\right),\ \frac{\partial}{\partial x}(x^3yz^2) - \frac{\partial}{\partial y}(xy^2z^3) \right\rangle$$

$$= \langle 3x^2y^2z - 2x^3yz,\ 3xy^2z^2 - 2xy^3z,\ 3x^2yz^2 - 2xyz^3 \rangle$$

we know if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi = \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle$

what is the curl of such a vector field?

In other words, what is curl$(\vec{\nabla}\phi)$ or curl(grad ϕ) ?

$$\text{curl}(\vec{\nabla}\phi) = \vec{\nabla} \times \vec{\nabla}\phi$$

$$= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \langle \underbrace{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}}_{\phi_{zy} - \phi_{yz} = 0}, -\underbrace{\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x} \right)}_{\phi_{xz} - \phi_{zx} = 0}, \underbrace{\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}}_{\phi_{xy} - \phi_{yx} = 0} \rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

$\boxed{\text{the curl of a conservative vector field is the zero vector}}$

if we take the dot product of $\vec{\nabla}$ with a vector field $\vec{F}$, we get the divergence of $\vec{F}$

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}}$$

in 2D, $\vec{F} = \langle f, g, 0\rangle$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle f, g, 0\rangle$$

$$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial 0}{\partial z} = f_x + g_y$$

this is how we defined divergence last time

<u>example</u> $\vec{F} = \langle xy^2z^3, x^3yz^2, x^2y^3z\rangle$

$$\operatorname{div}\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle xy^2z^3, x^3yz^2, x^2y^3z\rangle$$

$$= \frac{\partial}{\partial x}(xy^2z^3) + \frac{\partial}{\partial y}(x^3yz^2) + \frac{\partial}{\partial z}(x^2y^3z)$$

$$= \boxed{y^2z^3 + x^3z^2 + x^2y^3}$$

scalar!

(curl is a vector)

curl is easy in the picture of $\vec{F}$ → look for rotation

what about the divergence? Can we see it?

$\vec{F} = \langle f, g \rangle$

$\operatorname{div} \vec{F} = f_x + g_y$

same idea: how y-component changes as we move up

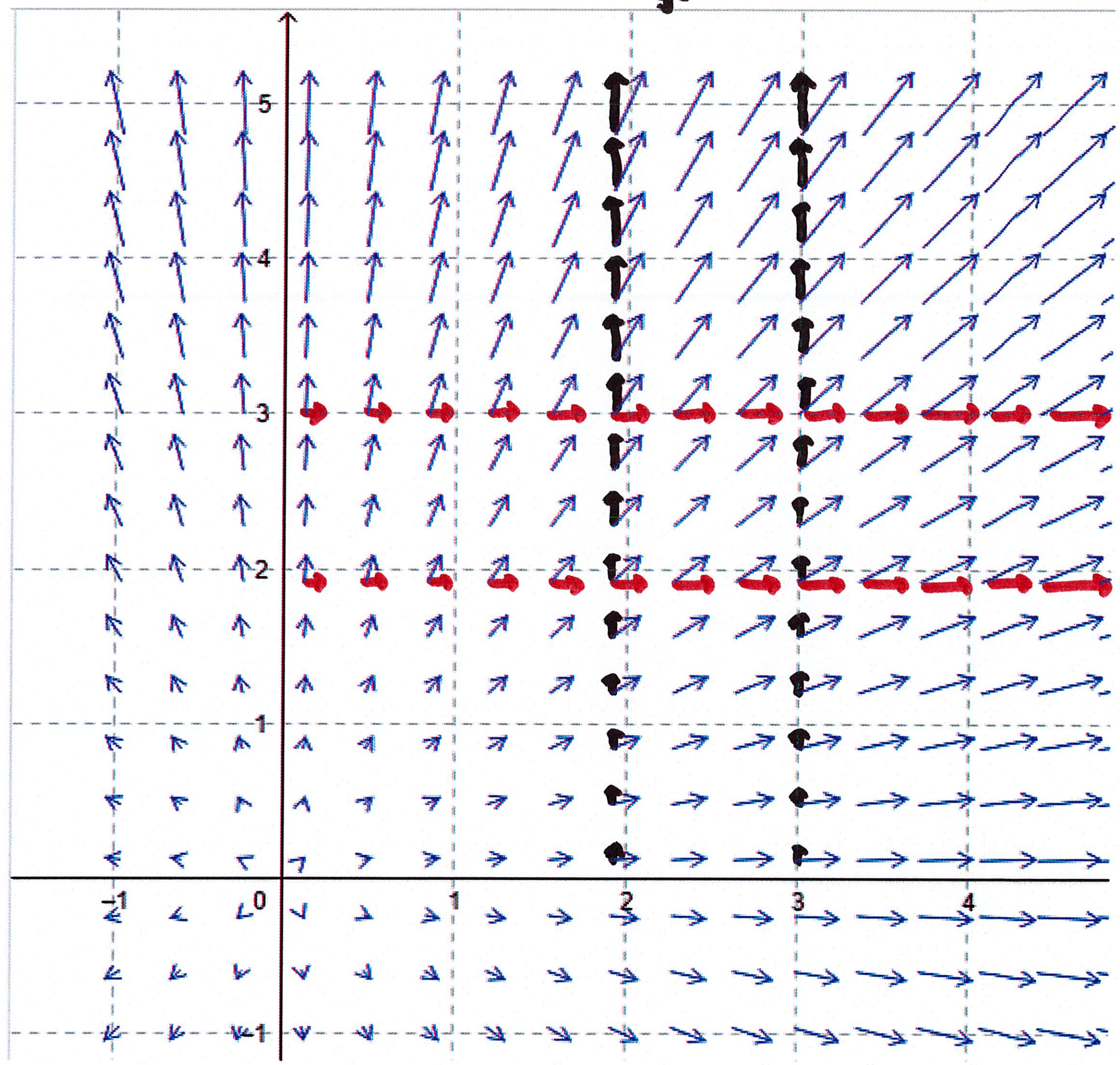
the rate of change of f (x-component of $\vec{F}$) as x increases

Another way: does the x-component get bigger or small as we move to the right?

div $\vec{F} = f_x + g_y$ what is the sign of div $\vec{F}$ of this $\vec{F}$?

y-component gets longer as we move up

so, $g_y > 0$



notice the x-component gets longer as we move right

so $f_x > 0$

so, for this vector field, div $\vec{F} = \underset{>0}{f_x} + \underset{>0}{g_y} > 0$