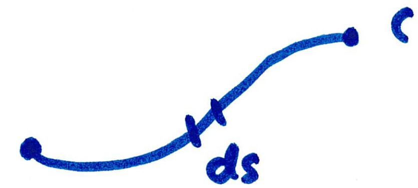
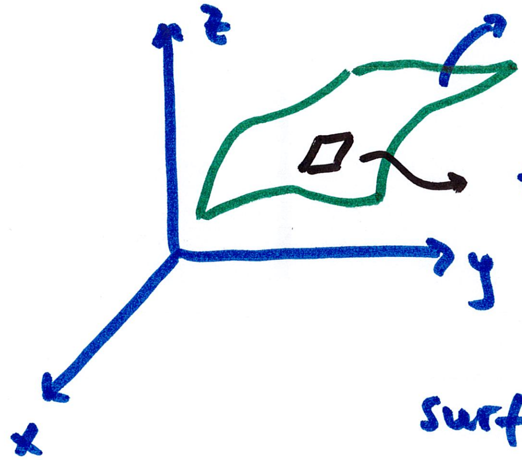
17.6 Surface Integrals (part 1)

line integral : $\int_C f(x,y,z)\,ds$



accumulation of $f(x,y,z)$ along curve C

Surface integral: accumulation of $f(x,y,z)$ all over a surface



call the surface S (capital S)

tiny piece of S, call this dS (captial S)

surface integral looks like : $\iint_S f(x,y,z)\,dS$

to compute a line integral $\int_C f(x,y,z)\,ds$ we need to parametrize the curve C

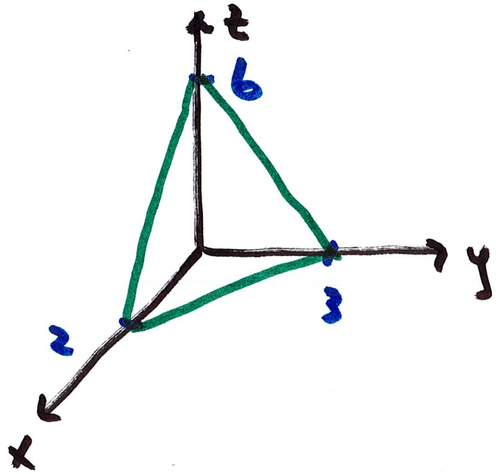
example: line $(-1,1)$ to $(2,4)$ along $y=x^2$

$\vec{r}(t) = \langle t, t^2 \rangle \qquad -1 \le t \le 2$

surface S: one more dimension than a curve C (one parameter t)
surface needs two

$\vec{r}(\underbrace{u,v}_{\text{parameters}})$ specify domain of u, v

example Parametrize the part of the surface $3x+2y+z=6$ in the first octant



parametrize: how to ~~location~~ locate each point on that surface

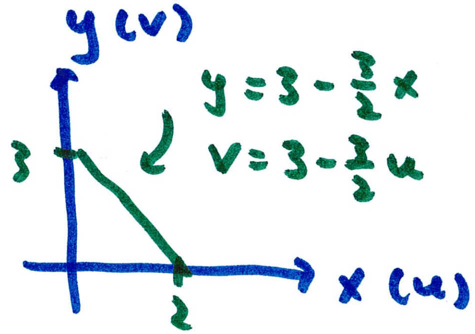
for example, if we know x, y then $z = 6-3x-2y$

knowing two we know the third

therefore, we will pick two of the variables to be the parameters

one possible parametrization: let $u = x$, $v = y$ then $z = 6-3u-2v$

so, this plane is $\vec{r}(\underset{x}{u},\underset{y}{v}) = \langle \underset{x}{u}, \underset{y}{v}, 6-3\underset{x}{u}-2\underset{y}{v} \rangle$



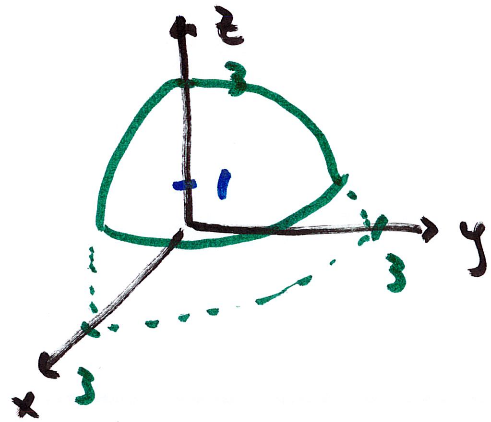
domain of u, v?

$0 \le u \le 2$, $\quad 0 \le v \le 3 - \frac{3}{2}u$

parametrization of this surface

$$\boxed{\begin{gathered}\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle \\ 0 \le u \le 2, \quad 0 \le v \le 3-\tfrac{3}{2}u\end{gathered}}$$

example $x^2+y^2+z^2=9$ in the first octant with $1 \le z \le 3$

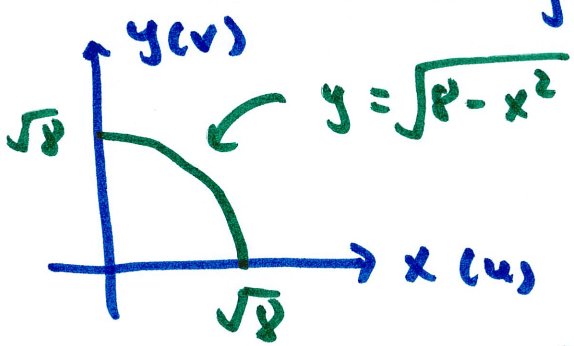


one possible way: parametrize like in last example

$u=x$, $v=y$, then $z=\sqrt{9-u^2-v^2}$

bounds for u, v

at $z=1 \rightarrow x^2+y^2=8$ circle radius $\sqrt{8}$



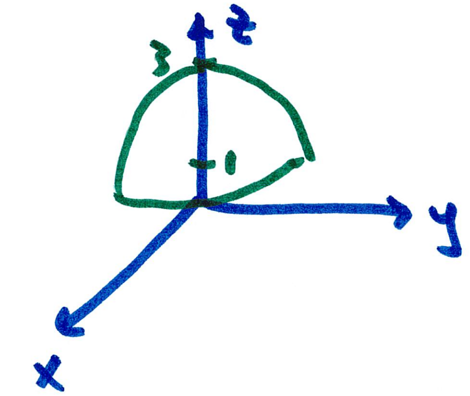
$0 \le x \le \sqrt{8}$

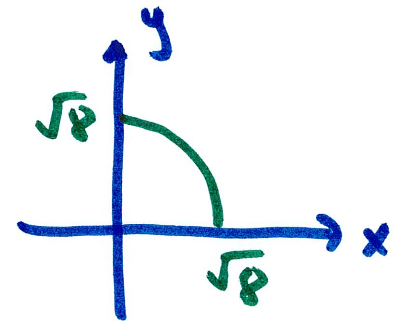
$0 \le y \le \sqrt{8-x^2}$

parametrization:

$$\boxed{\begin{array}{l} \vec{r}(u,v) = \langle u, v, \sqrt{9-u^2-v^2} \rangle \\ 0 \le u \le \sqrt{8}, \quad 0 \le v \le \sqrt{8-u^2} \end{array}}$$

another possibility: parametrization in cylindrical





$0 \le r \le \sqrt{8}$

$0 \le \theta \le \pi/2$

$1 \le z \le \sqrt{9-r^2}$

cylindrical : $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$ 3 variables: r, θ, z choose two to be parameters

let $u = r$, $v = \theta$

$0 \le u \le \sqrt{8}$, $0 \le v \le \pi/2$

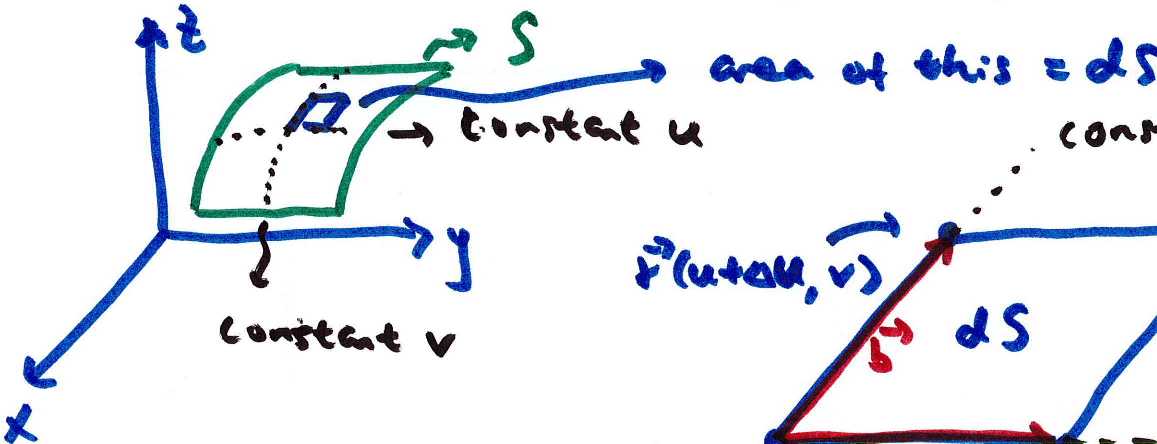
$$\boxed{\begin{gathered} \vec{r}(u,v) = \langle \underbrace{u\cos v}_{x = r\cos\theta}, \underbrace{u\sin v}_{y = r\sin\theta}, \underbrace{\sqrt{9-u^2}}_{z=\sqrt{9-r^2}} \rangle \\ 0 \le u \le \sqrt{8}, \quad 0 \le v \le \pi/2 \end{gathered}}$$

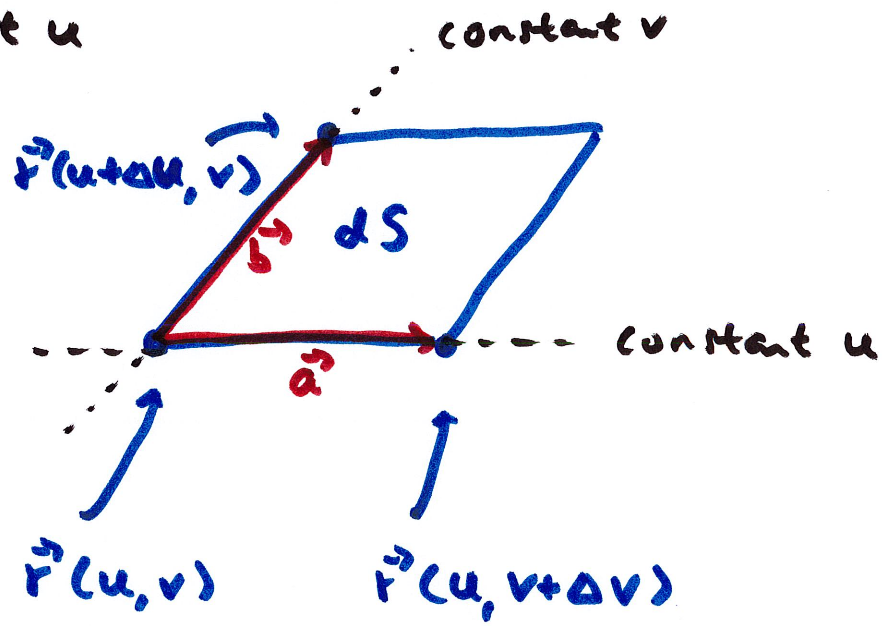
Surface integral: $\iint_S f(x,y,z)\,\underbrace{dS}_{\text{how to write this?}}$

line integral: $\int_C f(x,y,z)\,ds$

$ds = |\vec{r}\,'|\,dt$
length of small segment

in surface integral: dS is area of
a small patch of S





dS is a parallelogram so its area is $|\vec{a} \times \vec{b}|$

recall $\dfrac{\partial f}{\partial x} = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$

so, $\cancel{\dfrac{\partial f}{\partial x} \approx}$ $\dfrac{\partial f}{\partial x} \cdot h \approx f(x+h, y) - f(x, y)$

or $f_x \cdot h \approx f(x+h, y) - f(x, y)$

use this to write out $\vec{a}$, $\vec{b}$

$\vec{a} = \vec{r}(u, v+\Delta v) - \vec{r}(u, v) \longrightarrow \vec{a} = \vec{r}_v \Delta v$

$\vec{b} = \vec{r}(u+\Delta u, v) - \vec{r}(u, v) \longrightarrow \vec{b} = \vec{r}_u \Delta u$

$\cancel{ds = |\vec{r}_u \times \vec{r}_v|}$ $dS = |\vec{a} \times \vec{b}| = |\vec{r}_v \Delta v \times \vec{r}_u \Delta u|$

$= |\vec{r}_u \times \vec{r}_v| \Delta u \, dv$

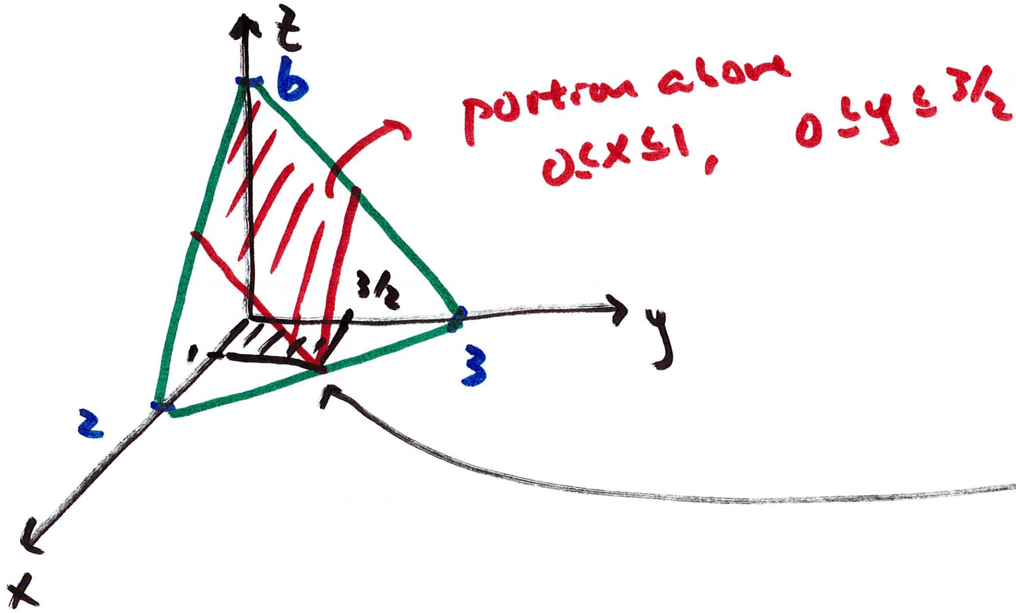
then $\Delta u \to du$, $\Delta v \to dv$ so, $\boxed{dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv}$

use this in surface integral

$\iint\limits_S f(x, y, z) \, dS$

↳ if $f = 1$, $\iint\limits_S dS$ = surface area

example $\iint_S (x+y)\,dS$ S: part of the plane $3x+2y+z=6$ in the first octant above $0\le x\le 1,\ 0\le y\le 3/2$



we only want the portion of plane above this

first, parameterize S: given bounds for x, y, so use them as parameters u, v

$u=x,\quad v=y,\quad z=6-3u-2v$

$$\boxed{\begin{gathered}\vec{r}(u,v)=\langle u,\ v,\ 6-3u-2v\rangle\\ 0\le u\le 1,\qquad 0\le v\le 3/2\end{gathered}}$$

$$\left.\begin{aligned}\vec{r}_u&=\langle 1,\ 0,\ -3\rangle\\ \vec{r}_v&=\langle 0,\ 1,\ -2\rangle\end{aligned}\right\}\ |\vec{r}_u\times\vec{r}_v|$$

$=|\langle 3,2,1\rangle|$

$=\sqrt{14}$

then $dS=|\vec{r}_u\times\vec{r}_v|\,du\,dv$

so, $\boxed{dS=\sqrt{14}\,du\,dv}$

$$\iint_S (x+y)\, dS \qquad dS = \sqrt{14}\, \underbrace{du\, dv}_{\text{or } dv\, du \text{ as appropriate}}$$

$x = u$ from parametrization

$y = v$ from $\vec{r}(u,v)$

$$= \int_{0}^{3/2}\!\!\!{}_{v} \int_{0}^{1}\!\!\!{}_{u} \underbrace{(u+v)}_{x+y}\, \underbrace{\sqrt{14}\, du\, dv}_{dS}$$

$$= \cdots = \boxed{\frac{15\sqrt{14}}{8}}$$