17.6 Surface Integrals (part 3)

last time : $\iint_S g(x,y,z)\,dS = \underbrace{\iint_S g(u,v)\,|\vec{r}_u \times \vec{r}_v|\,dA}_{\text{works in any coordinate system}}$ ← dudv or dvdu

often we get $z = f(x,y)$ → surface with z explicitly stated as function of x and y in Cartesian.

what does $\iint_S g(u,v)\,|\vec{r}_u \times \vec{r}_v|\,dA$ look like in this case?

$z = f(x,y)$

let $\vec{r}(u,v) = \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$\vec{r}_u = \vec{r}_x = \langle 1, 0, f_x \rangle \quad \vec{r}_v = \vec{r}_y = \langle 0, 1, f_y \rangle$

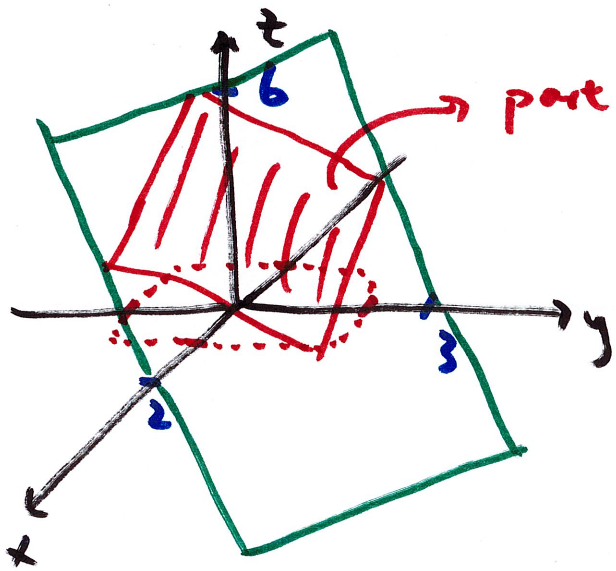
$\vec{r}_u \times \vec{r}_v = \vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle \qquad |\vec{r}_u \times \vec{r}_v| = \sqrt{1 + f_x^2 + f_y^2}$

$dS = |\vec{r}_u \times \vec{r}_v|\,dA = \sqrt{1 + f_x^2 + f_y^2}\,dA$

so, in the case where $z = f(x,y)$ in Cartesian, an explicitly formula

$$\boxed{\iint_R g(x,y,z)\sqrt{1 + f_x^2 + f_y^2}\,dA}$$

example $\iint_S (x+y)\,dS$ S: the plane $z = 6-3x-2y$ above the region $\underbrace{-1 \le x \le 1, \quad -2 \le y \le 2}$



part of plane above

here, we have z explicitly as function of x, y in Cartesian so the formula on last page can be used

$z = f(x,y) = 6-3x-2y$

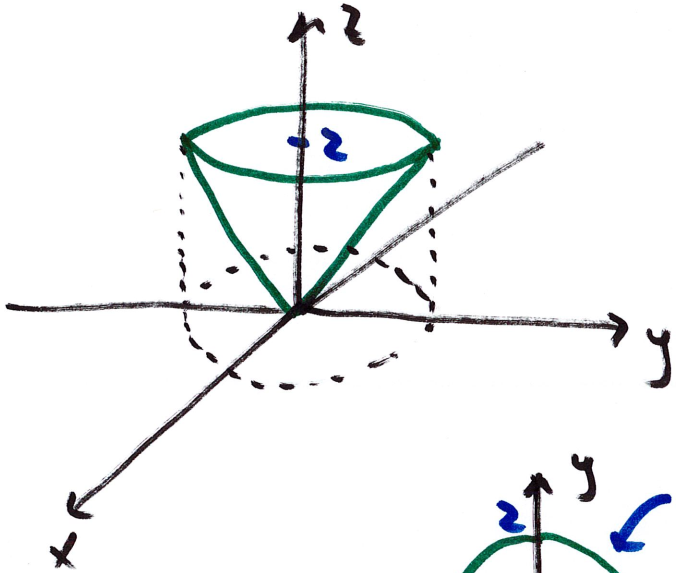
$f_x = -3, \quad f_y = -2$ so, $dS = \sqrt{1+f_x^2+f_y^2}\,dA = \sqrt{1+9+4}\,dA$

$= \sqrt{14}\,dA = \sqrt{14}\,dy\,dx$

$$\iint_S (x+y)\,dS = \int_{-1}^{1}\int_{-2}^{2} (x+y)\sqrt{14}\,dy\,dx = \cdots = \boxed{0}$$

$x \quad y$

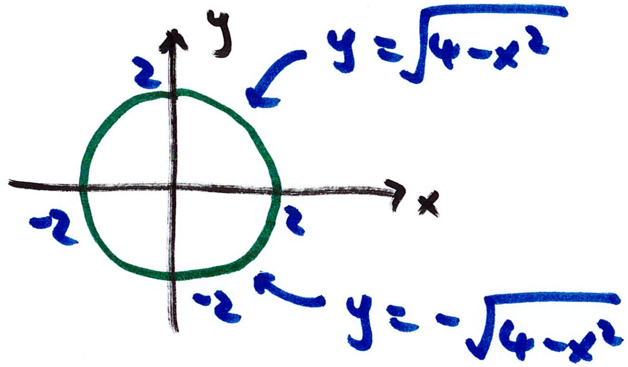
example Find the surface area of $z^2 = x^2+y^2$ $\quad 0 \le z \le 2$



$z = f(x,y) = \sqrt{x^2+y^2}$

the "shadow" of the cone of xy-plane is : $x^2+y^2 = (2)^2$ at $z=2$

circle radius 2



$-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

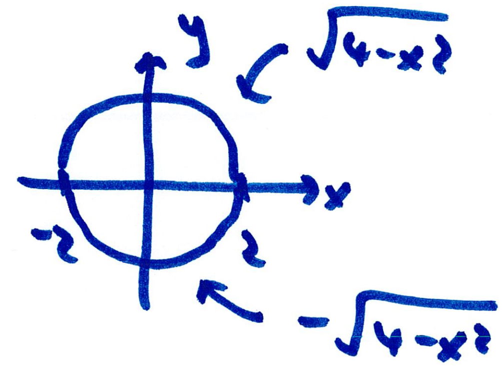
we can use $\iint_R g(x,y)\sqrt{1+f_x^2+f_y^2}\,dA$ again

$f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ $\quad f_y = \dfrac{y}{\sqrt{x^2+y^2}}$ $\quad$ so $dS = \sqrt{1+f_x^2+f_y^2}\,dA$

$$= \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\,dA = \sqrt{2}\,dA$$

Surface area: $\iint_R dS = \iint_R \sqrt{1+f_x^2+f_y^2}\, dA$ $\quad (g=1)$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\, dy\, dx$$



$$= \sqrt{2}\underbrace{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx}_{\text{area of circle radius 2}}$$

$$= \sqrt{2}\cdot \underbrace{\pi (2)^2}_{\pi r^2} = \boxed{4\sqrt{2}\pi}$$

alternative: change to polar

$0 \le r \le 2$

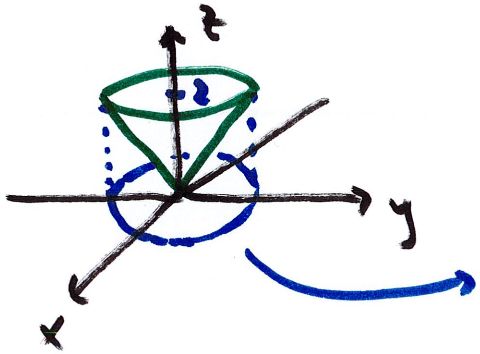
$0 \le \theta \le 2\pi$

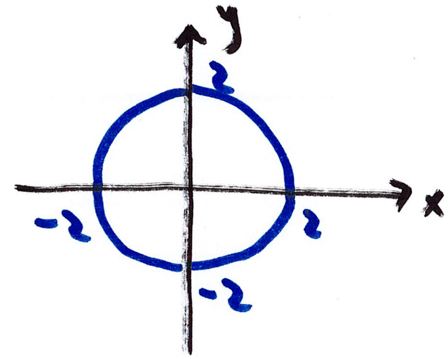
$$\sqrt{2}\int_0^{2\pi}\int_0^{2} \underbrace{r\, dr\, d\theta}_{dA} = \dots = \boxed{4\sqrt{2}\pi}$$

if we parametrized in a certain coord system (Cartesian in this example) and then change to another one, we need to manually adjust dA (e.g. $r\,dr\,d\theta$ in polar) but if we started the parametrization in ANY coordinate system, we do NOT need to manually adjust dA. $|\vec{r}_u \times \vec{r}_v|$ will automatically provide everything we need.

example (same example) Surface area of $z^2 = x^2 + y^2$, $0 \le z \le 2$

$z = \sqrt{x^2+y^2} = \sqrt{r^2} = r$





now let's choose to parametrize in cylindrical

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

let $u = r$, $v = \theta$

bounds: $0 \le u \le 2$
$0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u \rangle$

$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u \rangle$

$|\vec{r}_u \times \vec{r}_v| = \sqrt{2}\,u$

so $dS = |\vec{r}_u \times \vec{r}_v|\, dA = \sqrt{2}\, \underline{\underline{u}}\, dA$ ← $du\,dv$ or $dv\,du$

↳ in our parametrization, $u = r$
notice how $|\vec{r}_u \times \vec{r}_v|$ automatically supplied the extra r in cylindrical

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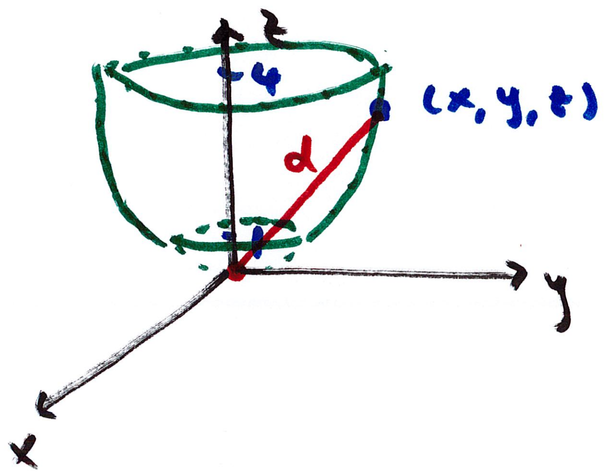
we can calculate the average value of $f(x,y,z)$ on S by doing something very similar to how we calculate the average w/ a double integral

$$f_{avg} = \frac{\iint_S f(x,y,z)\, dS}{\iint_S dS}$$

why? $\iint_S f(x,y,z)\, dS = f_{avg} \cdot \iint_S dS$

the accumulation of $f(x,y,z)$ over S is the same as if $f(x,y,z)$ is replaced with its average.

example Find the average distance of the points on $z = x^2 + y^2$ from the origin $1 \le z \le 4$



distance of (x, y, z) from origin is

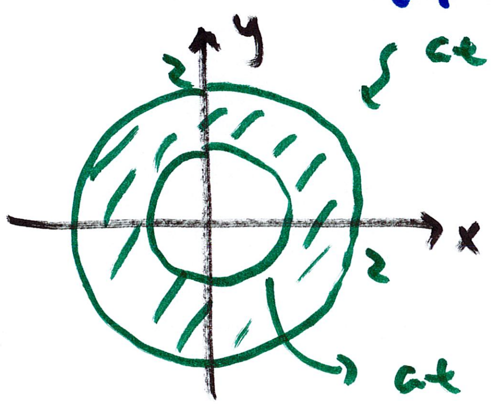
$$d = \sqrt{x^2 + y^2 + z^2}$$

what is the average distance of all points on the paraboloid

before parametrization, decide the coord. sys.

here, cylindrical looks good

project onto xy-plane



at $z = 4 \rightarrow 4 = x^2 + y^2$
radius 2

at $z = 1 \rightarrow 1 = x^2 + y^2$
radius 1

let $u = r$, $v = \theta$

$1 \le u \le 2$

$0 \le v \le 2\pi$

$z = x^2 + y^2 = r^2 = u^2$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u^2 \rangle$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 2u \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \cdots = u\sqrt{1+4u^2}$$

$$f_{avg} = \frac{\iint_S f\,dS}{\iint_S dS}$$

$$d = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+(r^2)^2} = \sqrt{r^2+r^4} = \sqrt{u^2+u^4}$$

$$\underbrace{\iint_S f}_{\text{distance}}\,dS = \int_0^{2\pi}\int_1^2 \sqrt{u^2+u^4}\cdot \underbrace{u\sqrt{1+4u^2}\,du\,dv}_{dS} = \cdots = 2\pi(15.21)$$

$$\iint_S dS = \int_0^{2\pi}\int_1^2 u\sqrt{1+4u^2}\,du\,dv = \cdots 2\pi(4.91)$$

so, $$f_{avg} = \frac{\iint_S f\,dS}{\iint_S dS} = \frac{2\pi(15.21)}{2\pi(4.91)} \approx 3.1$$

this is the average distance of points on the paraboloid from origin

Exam Advisory Grades

- Exam 2 average: 58
- Advisory grades of the two exams combined:
  - A: 168
  - B: 136
  - C: 104
  - D: 88
- These are based on the two exams alone. Homework and quizzes are not factored into these.
- Course grade will be assigned as described in the Course Ground Rules.