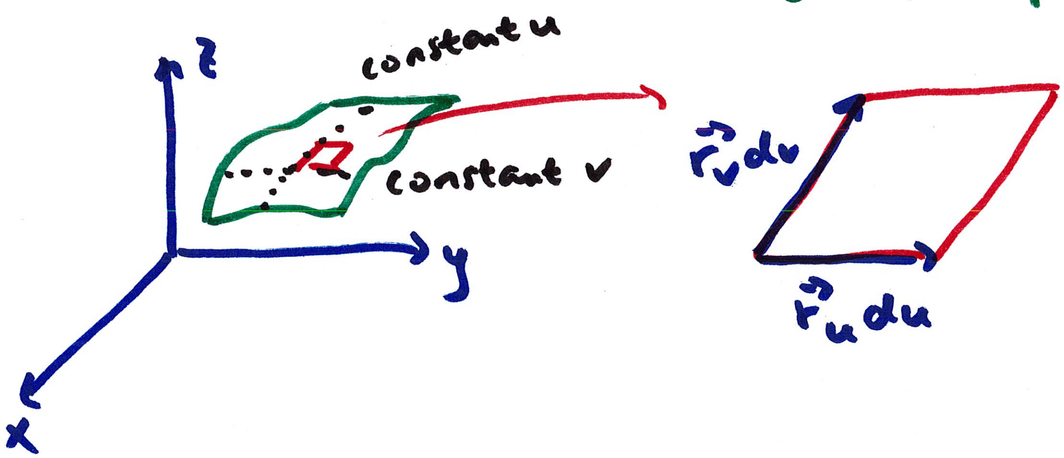
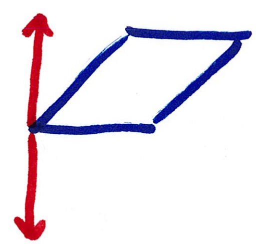
17.6 Surface Integrals (part 3)

last two times: surface integral in scalar field

$$\iint_S f(x,y,z)\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,dA}_{dS \text{ : area of small patch of surface } S}$$



$\vec{r}_u \times \vec{r}_v$ is the normal vector



$\vec{r}_v \times \vec{r}_u$ is also a normal vector but in opposite direction

in surface integral in scalar field, the order is irrelevant

$$|\vec{r}_u \times \vec{r}_v| = |\vec{r}_v \times \vec{r}_u|$$

but in a vector field, the direction of normal is important which do we choose?

by convention, we choose the upward or outward pointing normal

→ we call the surface S positively-oriented

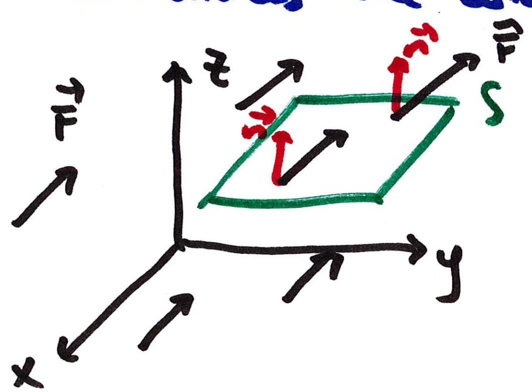
Surface integral in vector field: $$\iint_S \vec{F}\cdot \underbrace{d\vec{S}}_{\text{oriented surface}}$$

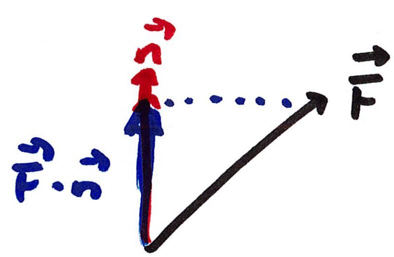
$$= \iint_S \vec{F}\cdot\vec{n}\, dS$$

unit normal with the correct orientation

this integral is often called the Flux Integral (for surface)

it accumulates the component of $\vec{F}$ in the same direction as $\vec{n}$





$\iint_S \vec{F}\cdot d\vec{S}$ gives us the amount of vector field flowing through S.

useful form of $\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_S \vec{F} \cdot \vec{n} \, dS$

$$\vec{n} = \frac{\vec{r_u} \times \vec{r_v}}{|\vec{r_u} \times \vec{r_v}|} \quad \text{or} \quad \frac{\vec{r_v} \times \vec{r_u}}{|\vec{r_v} \times \vec{r_u}|}$$ whichever is in the right direction

$$dS = |\vec{r_u} \times \vec{r_v}| \, dA = |\vec{r_v} \times \vec{r_u}| \, dA$$

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_S \vec{F} \cdot \vec{n} \, dS$$

$$= \iint\limits_S \vec{F} \cdot \frac{\vec{r_u} \times \vec{r_v}}{\cancel{|\vec{r_u} \times \vec{r_v}|}} \cancel{|\vec{r_u} \times \vec{r_v}|} \, dA = \boxed{\iint\limits_R \vec{F} \cdot \underbrace{(\vec{r_u} \times \vec{r_v})} \, dA}$$

or $\vec{r_v} \times \vec{r_u}$ whichever is in the "positive" direction

alternative form for $z = z(x,y)$

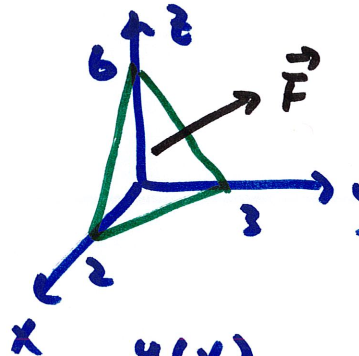
$\vec{F} = \langle f, g, h \rangle$

$$\iint\limits_S \vec{F} \cdot d\vec{S} = \iint\limits_R (-f z_x - g z_y + h) \, dA$$

example $\vec{F} = \langle x, y, z \rangle$ S: plane $3x+2y+z=6$ in the first octant

normal vector is positive upward

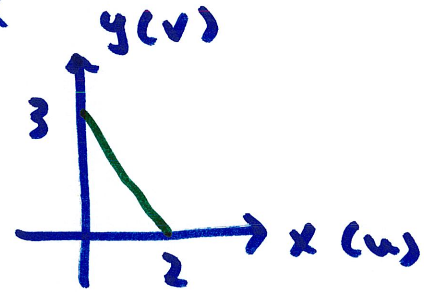
(if not stated, assume upward or outward)



first step: parametrize S

let $u = x$, $v = y$, $z = 6 - 3x - 2y$

$z = 6 - 3u - 2v$



$0 \le u \le 2$

$0 \le v \le 3 - \frac{3}{2}u$

$\vec{F}(u,v) = \langle u, v, 6-3u-2v \rangle$

$\vec{r}_u = \langle 1, 0, -3 \rangle$ $\vec{r}_v = \langle 0, 1, -2 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$ is this upward

yes, because z-component is positive

if not, reverse order of cross-product

rewrite $\vec{F}$ using x, y, z of parametrization

$$\vec{F} = \langle x, y, z \rangle = \langle u, v, 6-3u-2v \rangle$$

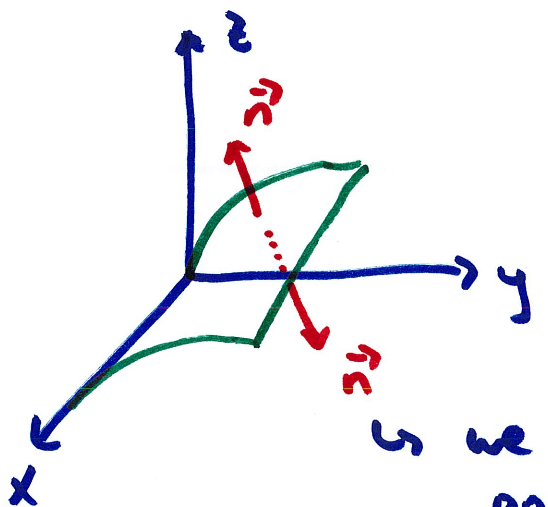
$$\iint_R \vec{F} \cdot (\vec{r_u} \times \vec{r_v})\, dA = \int_{\underset{u}{0}}^{2} \int_{\underset{v}{0}}^{3-\frac{3}{2}u} \underbrace{\langle u, v, 6-3u-2v \rangle}_{\vec{F}} \cdot \underbrace{\langle 3, 2, 1 \rangle}_{\vec{r_u} \times \vec{r_v}} \underbrace{dv\,du}_{dA}$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}u} 6\, dv\,du = \cdots = \boxed{18}$$

<u>example</u> $\vec{F} = \langle -y, x, 1 \rangle$

S: cylinder $y = z^2$, $0 \le x \le 3$, $0 \le z \le 1$

normal is positive toward the positive y-axis



↳ we want this normal because it points toward positive y-axis

parametrize S: $y = z^2$ we ~~gave~~ are given
$0 \le x \le 3$, $0 \le z \le 1$
so it's convenient to use x, z as parameters

so, choose $u = x$, $v = z$, then $y = z^2 = v^2$

$\vec{r}(u,v) = \langle u, v^2, v \rangle$ $\quad 0 \le u \le 3$
$0 \le v \le 1$

$\vec{r_u} = \langle 1, 0, 0 \rangle$ $\quad$ $\vec{r_v} = \langle 0, 2v, 1 \rangle$

$\vec{r_u} \times \vec{r_v} = \langle 0, -1, 2v \rangle$ $\quad$ is this "positive" (points towards positive y-axis)

no, because y-component < 0

fix by reversing order

want: $\vec{r_v} \times \vec{r_u} = \langle 0, 1, -2v \rangle$

$\vec{F} = \langle -y, x, 1 \rangle = \langle -v^2, u, 1 \rangle$

$$\iint_R \vec{F} \cdot (\vec{r_v} \times \vec{r_u})\, dA = \int_0^3 \int_0^1 \langle -v^2, u, 1 \rangle \cdot \langle 0, 1, -2v \rangle \, dv\, du$$

$$= \cdots = \boxed{\frac{3}{2}}$$

positive means the net flow of $\vec{F}$ is toward the direction of $\vec{n}$

<u>example</u> $\vec{F} = \dfrac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

S: sphere radius a

normal pointing <u>outward</u>

parametrize S: S is sphere, makes sense to do in spherical

$\rho =$ constant $= a$

so choose $u = \phi$, $v = \theta$

$0 \le u \le \pi \qquad 0 \le v \le 2\pi$

$$\vec{r}(u,v) = \langle \underbrace{a \sin u \cos v}_{\rho \sin\phi \cos\theta},\ a \sin u \sin v,\ a \cos u \rangle$$

$$\vec{r_u} = \langle a \cos u \cos v,\ a \cos u \sin v,\ -a \sin u \rangle$$

$$\vec{r_v} = \langle -a \sin u \sin v,\ a \sin u \cos v,\ 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle a^2 \sin^2 u \cos v,\ a^2 \sin^2 u \sin v,\ a^2 \cos u \sin u \rangle$$

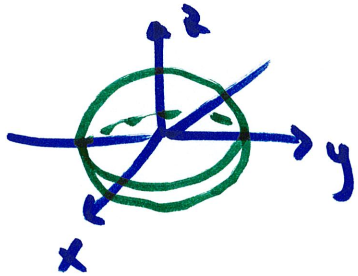
pointing out?

in $\underbrace{\text{first octant}}_{\sin u > 0,\ \cos v > 0}$, all ~~co~~ components ≥ 0

so seems right

then choose another octant to check

yes



$$\vec{F} = \frac{-\langle x, y, z\rangle}{(\underbrace{x^2+y^2+z^2}_{\rho^2 = a^2})^{3/2}} = \frac{-\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle}{a^3}$$

$$\int_{\underset{v}{0}}^{2\pi}\int_{\underset{u}{0}}^{\pi} \vec{F}\cdot(\underbrace{\vec{r_u}\times\vec{r_v}}_{\text{on last page}})\,du\,dv = \cdots = \boxed{-4\pi}$$