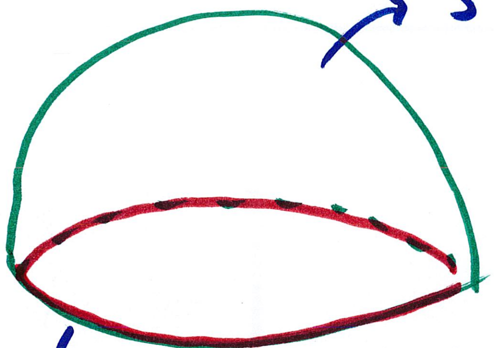
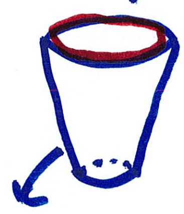
17.7 Stokes' Theorem (part 1)

Stokes' Theorem relates the surface integral of the curl of a vector field to a line integral along the boundary of the surface



S (surface)
open surface

C: the curve of the boundary (open part)



C

S

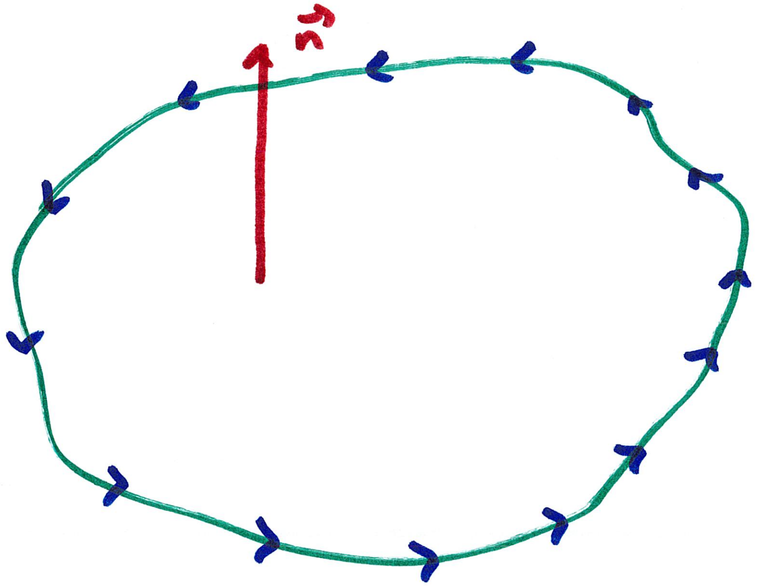
cup w/o lid

S : surface of cup

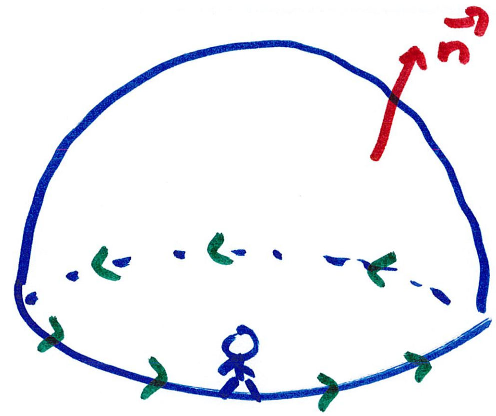
C : boundary (the open) part is the top

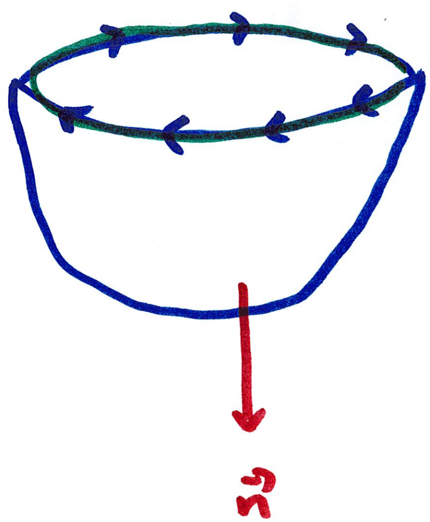
both the surface and the boundary curve are oriented

their orientations obey the right-hand rule



align thumb of right hand w/ the normal vector of S ($\vec{n}$) the direction the other fingers curl and point is the orientation of the boundary curve C





imagine walking along boundary w/ head toward the normal vector walk such that the enclosed area of C is on your <u>LEFT</u>

Stokes' Theorem

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

S: surface

C: boundary curve

$d\vec{S} = \vec{n}\,dS$

$= (\vec{r_u}\times\vec{r_v})\,du\,dv$ or

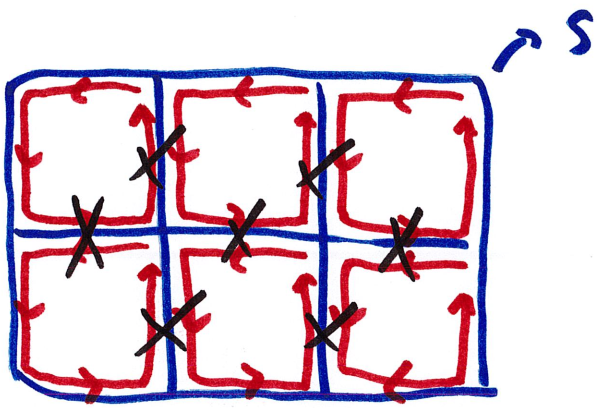
$(\vec{r_v}\times\vec{r_u})\,du\,dv$

whichever points the normal in the positive direction

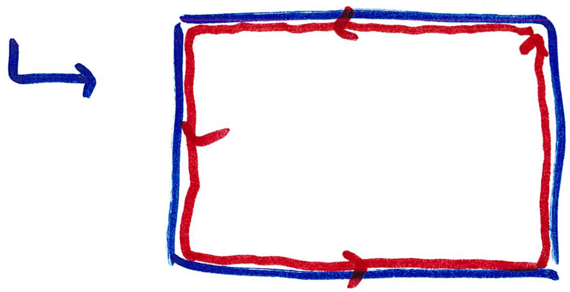
why is $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

the reason is actually the same as with Green's Theorem

Left side : $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ accumulates the curl (the little circulations) all over the surface



interior flows cancel out



which is what the right side $\oint_C \vec{F}\cdot d\vec{r}$ is doing

Stokes' is a more general version of Green's Theorem

(or Green's is a special case of Stokes')

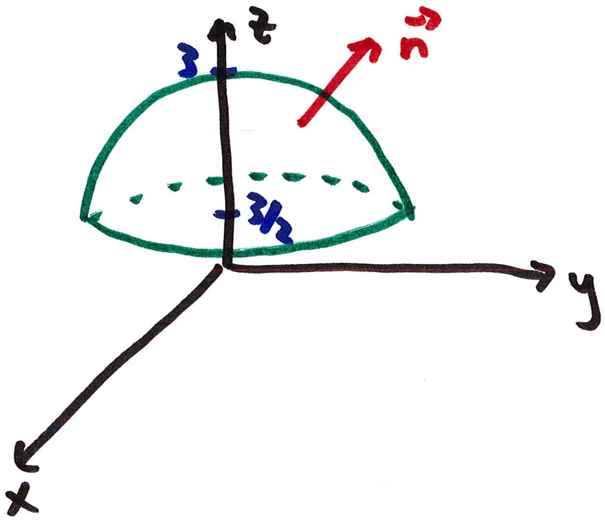
Green's : surface must be flat

Stokes' : surface can be flat or curved

<u>example</u> $\vec{F} = \langle y, -x, 0 \rangle$

S: sphere radius 3, $z \geq 3/2$, normal points outward.

verify Stokes' Theorem.



parametrize S: let's go with spherical variables: ρ, θ, ϕ

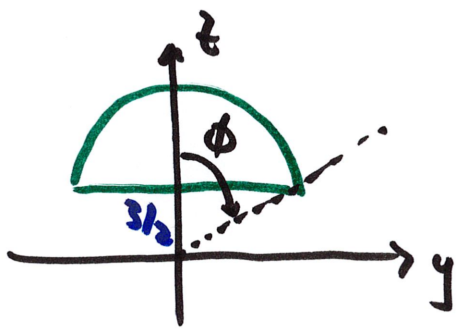
here, $\rho = 3$ constant

so choose θ, ϕ to be parameters

let $u = \phi$, $v = \theta$

$$\boxed{\vec{r}(u,v) = \langle \underbrace{3 \sin u \cos v}_{\rho \sin\phi \cos\theta}, 3 \sin u \sin v, 3 \cos u \rangle}$$

$$\boxed{0 \leq v \leq 2\pi}$$



for u: $z \geq 3/2 \rightarrow 3 \cos u \geq 3/2$

$\cos u \geq 1/2$

$$\boxed{0 \leq u \leq \pi/3}$$

$\vec{r}_u = \langle 3\cos u\cos v,\ \ 3\cos u\sin v,\ -3\sin u \rangle$

$\vec{r}_v = \langle -3\sin u\sin v,\ \ 3\sin u\cos v,\ \ 0 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\sin u\cos u \rangle$

is this <u>outward</u>? yes

pick a few places and check if $\hat{z}$ is ≥ 0 (here, outward is also equivalent to $\hat{z}$ up)

now do $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ $\quad \vec{F} = \langle y, -x, 0\rangle \quad \text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \cdots = \langle 0, 0, -2\rangle$

$$= \iint_R \text{curl}\,\vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

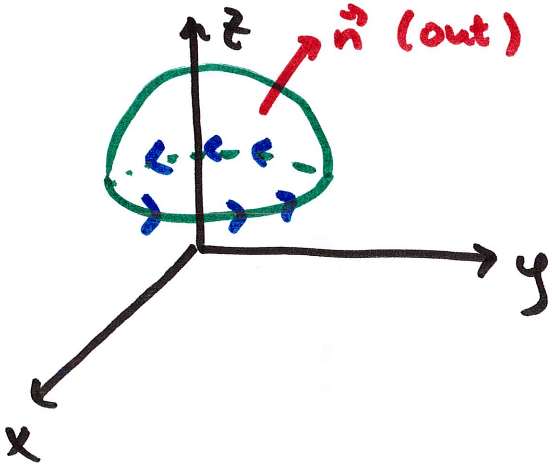
$$= \int_{\substack{0 \\ v}}^{2\pi}\int_{\substack{0 \\ u}}^{\pi/3} \langle 0,0,-2\rangle\cdot\langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\sin u\cos u\rangle\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/3} -18\sin u\cos u\,du\,dv = \cdots = \boxed{-27\pi/2}$$

Stokes': $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

let's see if $\oint_C \vec{F}\cdot d\vec{r}$ is easier

parametrize the boundary curve w/ the right orientation



$\vec{n}$ is out, by right-hand rule, C goes counterclockwise when viewed from above.

C : circle radius ?

$x^2+y^2+z^2=9$ C is at $z=3/2$

$x^2+y^2+\frac{9}{4}=9$ $x^2+y^2=\frac{27}{4}$

radius of C is $\frac{\sqrt{27}}{2}$

C: $\vec{r}(t)=\langle \frac{\sqrt{27}}{2}\cos t,\ \frac{\sqrt{27}}{2}\sin t,\ 3/2 \rangle$ $0\le t\le 2\pi$

$$\vec{F} = \langle y, -x, 0 \rangle = \left\langle \frac{\sqrt{27}}{2}\sin t, -\frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$$

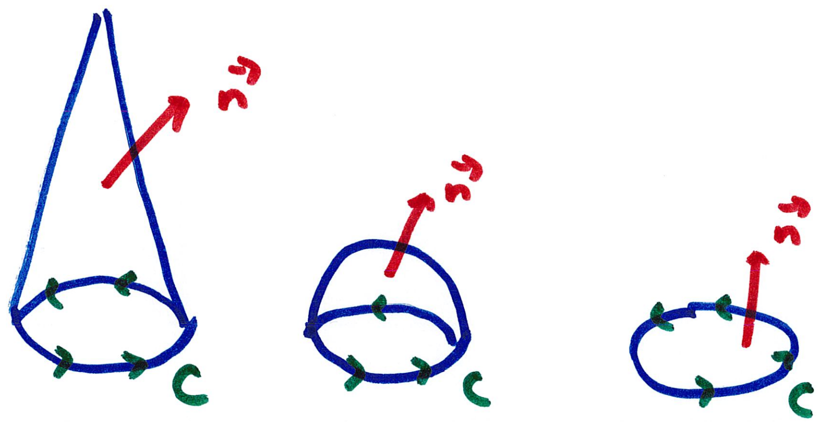
$$d\vec{r} = \vec{r}\,' dt = \left\langle -\frac{\sqrt{27}}{2}\sin t, \frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$$

$$\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\frac{27}{4}\sin^2 t - \frac{27}{4}\cos^2 t \, dt$$

$$= -\frac{27}{4}\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = -\frac{27}{4}\cdot 2\pi = \boxed{-\frac{27}{2}\pi}$$

Stokes' Theorem : $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

this means that two surfaces with the same boundary C must have the same $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$



all of these share the same boundary C

so $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ is the same for all!

in practice, this means we can replace S with a simpler one w/ same C if we want to compute the surface integral