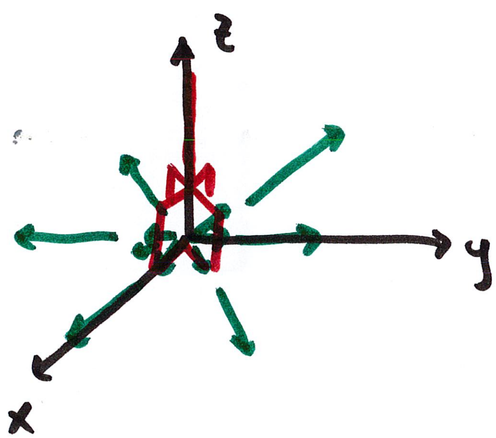
# 17.7 Stokes' Theorem (part 2)

what does $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$ mean physically?

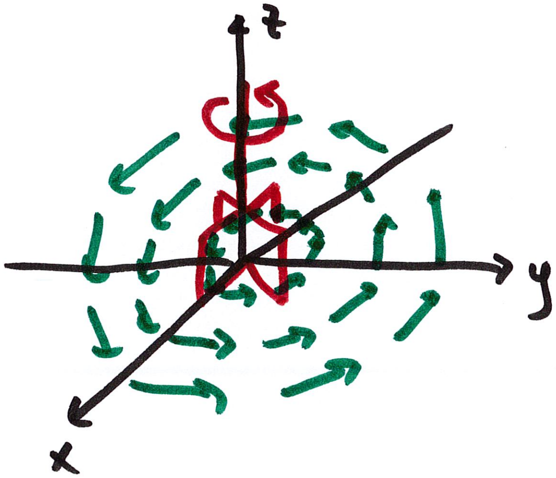
let's take another look at $\operatorname{curl} \vec{F}$

$\vec{F} = \langle x, y, 0 \rangle$



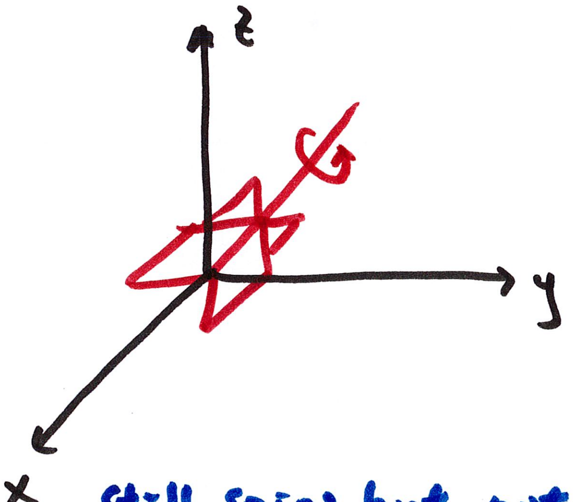

if a paddle wheel with axis along z-axis is placed at the origin

we don't expect it to spin since the vector field pushes all fins radially out now look at $\vec{F} = \langle -y, x, 0 \rangle$

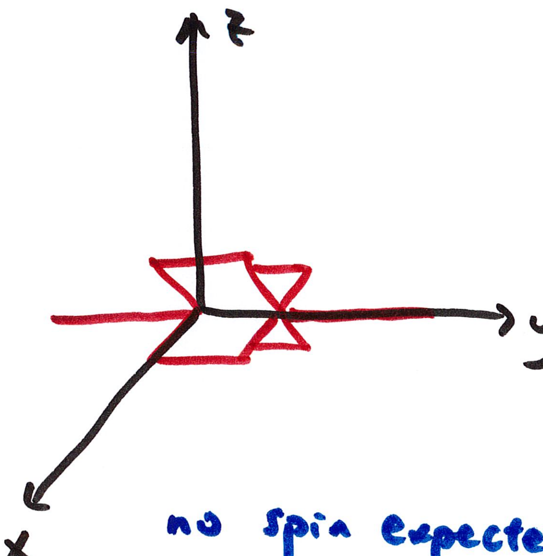



now we expect the paddle wheel w/ axis along z-axis to spin

if we tilt the axis of the paddle wheel, the spin changes




still spins but not as fast




no spin expected

the first vector field $\vec{F} = \langle x, y, 0 \rangle$ curl $\vec{F} = \vec{\nabla} \times \vec{F} = \langle 0, 0, 0 \rangle = \vec{0}$

the second one $\vec{F} = \langle -y, x, 0 \rangle$ curl $\vec{F} = \underbrace{\langle 0, 0, 2 \rangle}$

these numbers tell us that to get the maximum paddle wheel spin, we want to align its axis with z-axis (where the nonzero number is)

if aligned w/ x or y axis (curl is 0 in those) we get no spin.

$\vec{F} = \langle 5 - z^2, 0, 0 \rangle$

curl $\vec{F} = \underbrace{\langle 0, -2z, 0 \rangle}$

to get max spin, place axis of paddle wheel along y-axis where the curl is the greatest

and the higher $|z|$ is, the faster the spin because $|-2z|$ gets bigger

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \underbrace{\iint_S \operatorname{curl}\vec{F}\cdot \underline{\vec{n}}\, dS}$$

Stokes' : $\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

this accumulates the spin of a paddle wheel with its axis along the normal vector of the surface

<u>example</u> $S$ : upper half of $\underline{z^2 = a^2(1-x^2-y^2)}$ $\quad a$ : some number

$\vec{n}$ is positive upward

shape?

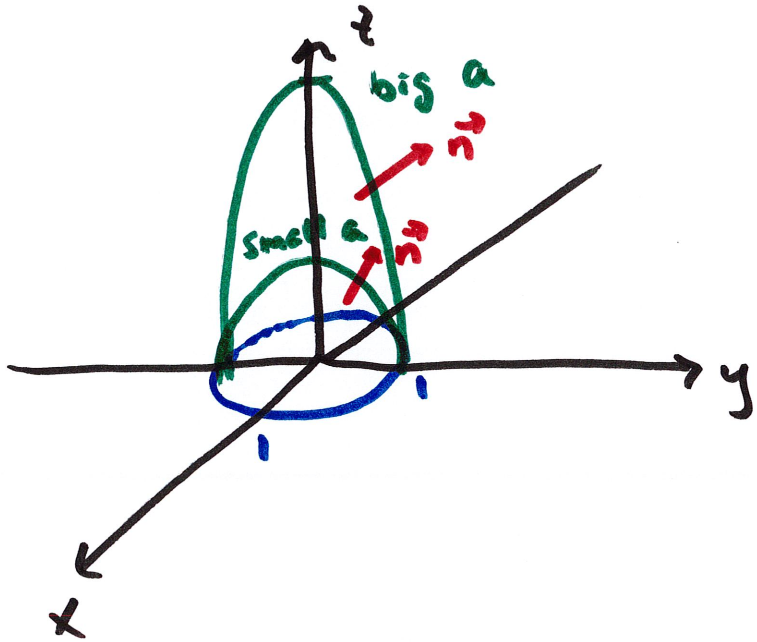
$\vec{F} = \langle x-y,\ y+z,\ z-x\rangle$

find $a$ such that $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$ is maximized.

shape of $z^2 = a^2(1-x^2-y^2)$ ?

$$\frac{z^2}{a^2} = 1-x^2-y^2$$

$$x^2+y^2+\frac{z^2}{a^2} = 1 \qquad \text{ellipsoid}$$




as $a$ changes, $\vec{n}$ changes orientation

so we expect $\displaystyle\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$

to depend on $a$.

let's calculate $\displaystyle\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \iint_S \text{curl}\,\vec{F}\cdot\underbrace{(\vec{r}_u\times\vec{r}_v)}_{\text{points in the positive direction}}dA$

surface : $z^2 = a^2(1-x^2-y^2)$

upper half : $z = a\sqrt{1-x^2-y^2}$

parametrize S. Coordinate system? Let's use cylindrical

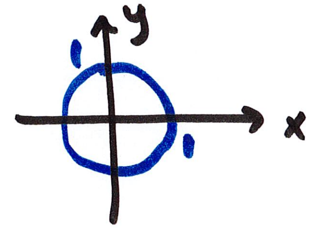
let $u=r$ $\quad v=\theta$

$x = r\cos\theta = u\cos v$

$y = r\sin\theta = u\sin v$

$z = a\sqrt{1-x^2-y^2} = a\sqrt{1-r^2} = a\sqrt{1-u^2}$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ a\sqrt{1-u^2} \rangle \quad 0 \le u \le 1,\ 0 \le v \le 2\pi$




$0 \le u \le 1$
$0 \le v \le 2\pi$

$$\vec{r}_u = \left\langle \cos v,\ \sin v,\ \frac{-au}{\sqrt{1-u^2}} \right\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \left\langle \frac{au^2\cos v}{\sqrt{1-u^2}},\ \frac{au^2\sin v}{\sqrt{1-u^2}},\ u \right\rangle$$

does this point in the positive (in this example, upward) direction?

yes, since the z-component is always between 0 and 1

$$\vec{F} = \langle x-y,\ y+z,\ z-x \rangle$$

$$\text{curl}\,\vec{F} = \langle 1,\ 1,\ 1 \rangle$$

$$\iint_S \text{curl}\,\vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\,dA = \int_0^{2\pi}\int_0^1 \langle 1,1,1\rangle \cdot \left\langle \frac{au^2\cos v}{\sqrt{1-u^2}},\ \frac{au^2\sin v}{\sqrt{1-u^2}},\ u \right\rangle du\,dv$$

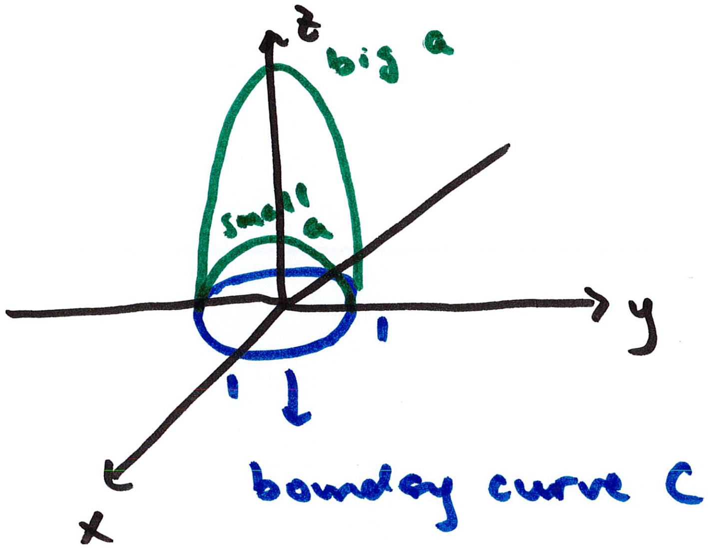
$$= \cdots = \boxed{\pi}$$

surprising, since it does not depend on a

this calculation is not terrible, but we could actually get it much faster by using Stokes' Theorem

Stokes' : $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

$C$ ↳ boundary curve




Since the boundary curve is not affected by a (it's always circle radius 1), Stokes' Theorem says the surface ~~ip~~ integral of the curl will not be affected by a.

Stokes' Theorem: $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$

$\iint_S \underline{\underline{\text{curl}}}\,\vec{F}\cdot d\vec{S}$

only use Stokes' if you are integrating the $\underline{\text{curl of } \vec{F}}$

DO $\underline{\underline{\text{NOT}}}$ use it for "regular" surface integrals $\iint_S \vec{F}\cdot d\vec{S}$