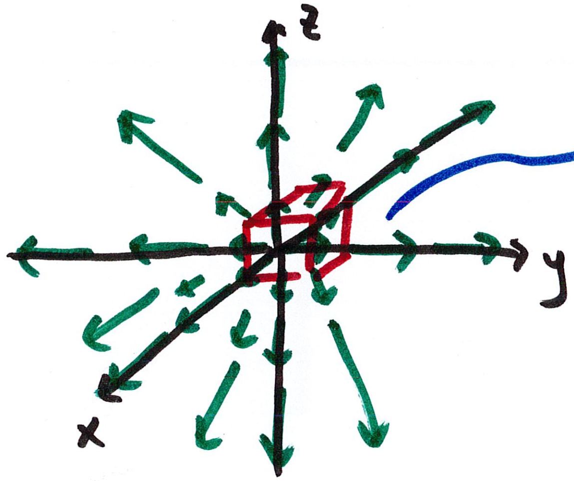
# 17.8 The Divergence Theorem (part 1)

divergence : $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \langle f, g, h \rangle = f_x + g_y + h_z$

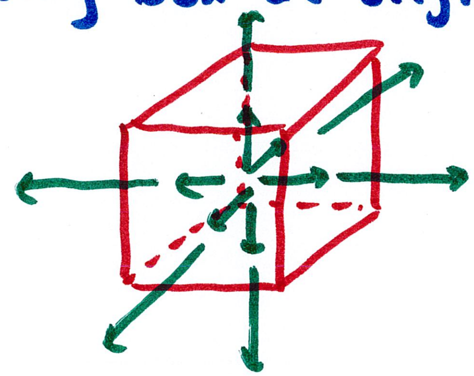
for example, $\vec{F} = \langle x, y, z \rangle$ $\quad \operatorname{div} \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$

↑ what does this mean?




outward radial flow

tiny box at origin



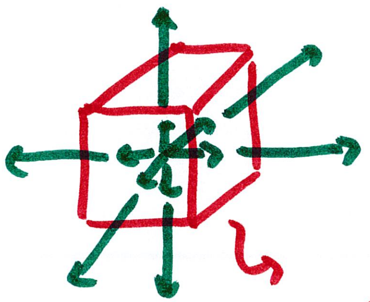
magnitude of $\vec{F}$ increases as it flows out, there is a stronger vector field coming out than inside

if the box is flexible, then this will cause the box to expand

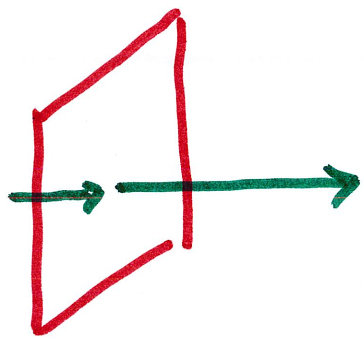
if $\operatorname{div} \vec{F} > 0 \rightarrow$ box expands

if $\operatorname{div} \vec{F} < 0 \rightarrow$ box shrink if the box is not flexible but the surfaces are porous (things flow through)

then the box will not grow or shrink



each face will have greater magnitude of $\vec{F}$ ~~out~~ leaving it than ~~entering~~ entering it



this gives a net flow outward on each surface (if $\operatorname{div}\vec{F} > 0$)

if $\operatorname{div}\vec{F} < 0$, opposite situation

difference in flow $\rightarrow$ <u>flux integral</u>

so, the divergence is related to change in volume and flux through surface

$\rightarrow$ <u>Divergence Theorem</u>

$$\underbrace{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS}_{\text{flux through } S} = \underbrace{\iiint_D \operatorname{div}\vec{F}\,dV}_{\text{accumulation of divergence inside volume}}$$

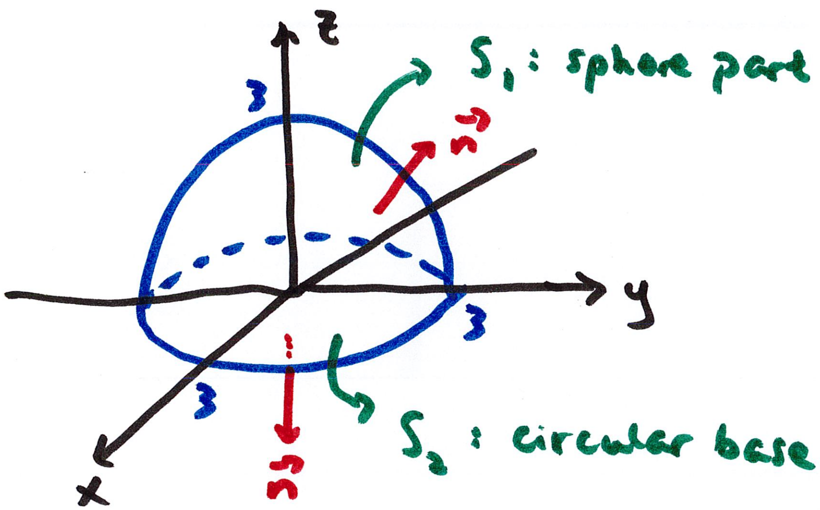
the theorem is primarily used with <u>closed</u> surface (no opening/hole)

**example** $\vec{F} = \langle x, y, z \rangle$

$S$ : upper half of sphere radius 3 including the base at $z = 0$

as usual, $\vec{n}$ is positive outward

Verify the Divergence Theorem




notice $\vec{n}$ outward means $\vec{n}$ upward on $S_1$
and downward on $S_2$

first, let's do $\iint_S \vec{F} \cdot \vec{n} \, dS$ (flux integral)

parametrize $S_1$ : sphere, so use spherical $u = \phi$, $v = \theta$

$$\vec{r}(u,v) = \langle \underbrace{3 \sin u \cos v}_{\rho \sin\phi \cos\theta}, \; 3 \sin u \sin v, \; 3 \cos u \rangle \qquad 0 \leq u \leq \pi/2, \quad 0 \leq v \leq 2\pi$$

$$\vec{r}_u = \langle 3 \cos u \cos v, \; 3 \cos u \sin v, \; -3 \sin u \rangle$$

$$\vec{r}_v = \langle -3 \sin u \sin v, \; 3 \sin u \cos v, \; 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 9 \sin^2 u \cos v, \; 9 \sin^2 u \sin v, \; 9 \cos u \sin v \rangle$$

direction correct? up? yes

$S_2$ : circle at $z=0$, so use polar $\quad u=r, \quad v=\theta$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ 0\rangle \qquad 0 \le u \le 3 \qquad 0 \le v \le 2\pi$$

$$\vec{r}_u = \langle \cos v,\ \sin v,\ 0\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0,\ 0,\ u\rangle$$

direction correct?
no, since this points up
we want it down on $S_2$
fix: swap cross order or sign
we want $\vec{r}_v \times \vec{r}_u = \langle 0, 0, -u\rangle$

now the flux integral $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$
or $\vec{r}_v \times \vec{r}_u$

$$\underbrace{\int_0^{2\pi}\int_0^{\pi/2} \underbrace{\langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle}_{\vec{F}\text{ using } x,\,y,\,z \text{ of } \vec{r}(u,v) \text{ for } S_1}\cdot\langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin u\rangle\,du\,dv}_{S_1}$$

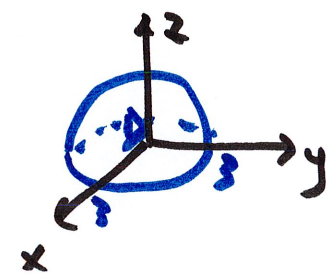
$$+\int_0^{2\pi}\int_0^{3} \overbrace{\langle u\cos v,\ u\sin v,\ 0\rangle}^{\vec{F}}\cdot\langle 0, 0, -u\rangle\,du\,dv = \int_0^{2\pi}\int_0^{\pi/2}(27\sin^3 u + 27\cos^2 u\sin u)\,du\,dv$$

$$= \int_0^{2\pi} \int_0^{\pi/2} 27 \sin u \, du \, dv = \boxed{54\pi}$$

the Divergence Theorem: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F} \, dV$

the whole thing can be replaced w/ $\iiint_D \operatorname{div} \vec{F} \, dV$

↳ enclosed volume




$\vec{F} = \langle x, y, z \rangle \quad \operatorname{div} \vec{F} = 3$

$$\iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D 3 \, dV = 3 \underbrace{\iiint_D dV}_{}$$

geometrically is

the volume of the upper half

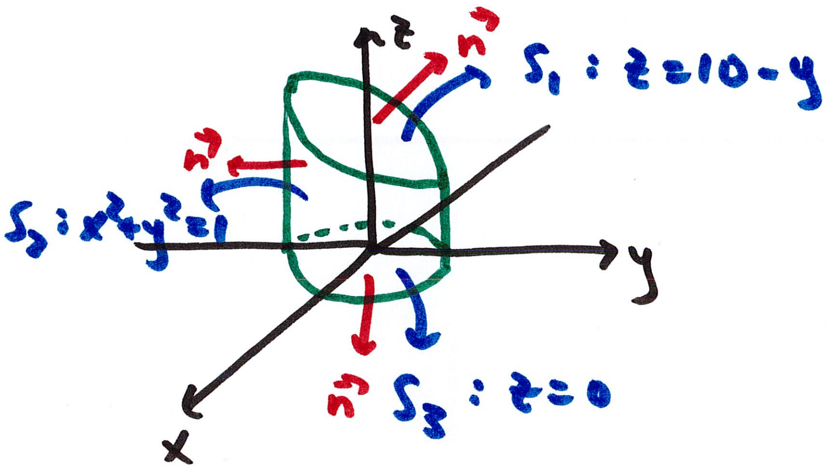
of sphere of radius 3

$$= 3 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi (3)^3 = 2\pi \cdot 27 = \boxed{54\pi}$$

<u>example</u> $\vec{F} = \langle -y^3z^2,\ x^4,\ 4xy^2 \rangle$

$S$ : surface of solid bounded by $\underbrace{x^2+y^2=1}_{\text{cylinder}}$, $\underbrace{z=10-y}_{\text{plane}}$, $\underbrace{z=0}_{\text{plane}}$

normal is positive outward




3 surfaces, 3 parametrizations,
3 surface integrals
so makes sense to use Divergence Theorem to do 1 volume integral instead

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle -y^3z^2,\ x^4,\ 4xy^2\rangle$$
$$= \frac{\partial}{\partial x}(-y^3z^2) + \frac{\partial}{\partial y}(x^4) + \frac{\partial}{\partial z}(4xy^2) = 0$$

$$\iiint_D 0\, dV = \boxed{0}$$