17.8 The Divergence Theorem (part 2)

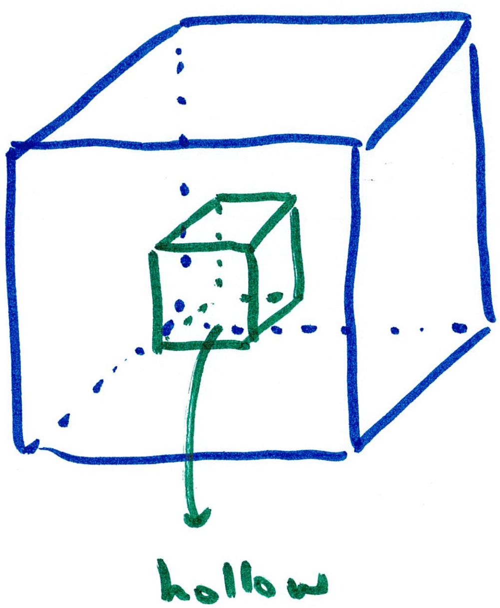
$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$$

D: space enclosed by S

$\vec{n}$ is assumed to be <u>outward</u>

if inward, flip sign : $-\iiint_D \operatorname{div}\vec{F}\,dV$

what if S has a hollow space inside, for example, a hollow cube or sphere?



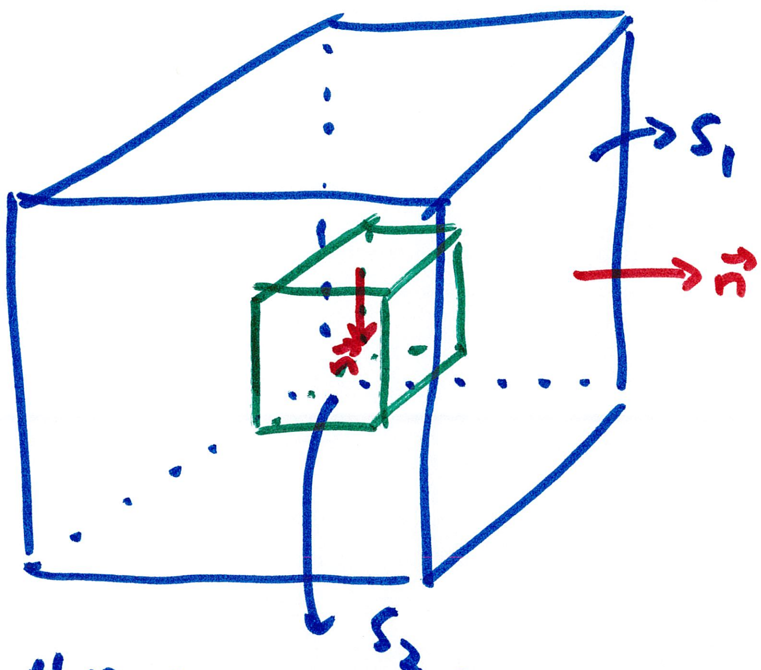
S: the surface that forms the hollow cube outside of blue and inside it that contains the space removed by the green

S_1: the outside surface (blue)

S_2: the inside surface (green)

D: volume between the cubes

normal vector, as usual is <u>outward-pointing</u> (pointing <u>away</u> from the volume enclosed by S)



outside: points out

inside: points <u>inward</u>

in this case, the flux integral is

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS$$

due to the inner cube having inward-pointing normal

then the Divergence Theorem says:

$$\boxed{\iint_{S_1} \vec{F}\cdot\vec{n}\,dS - \iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV}$$

space between cubes

let D_2 be the space inside inner cube and D_1 be the space inside the outer cube

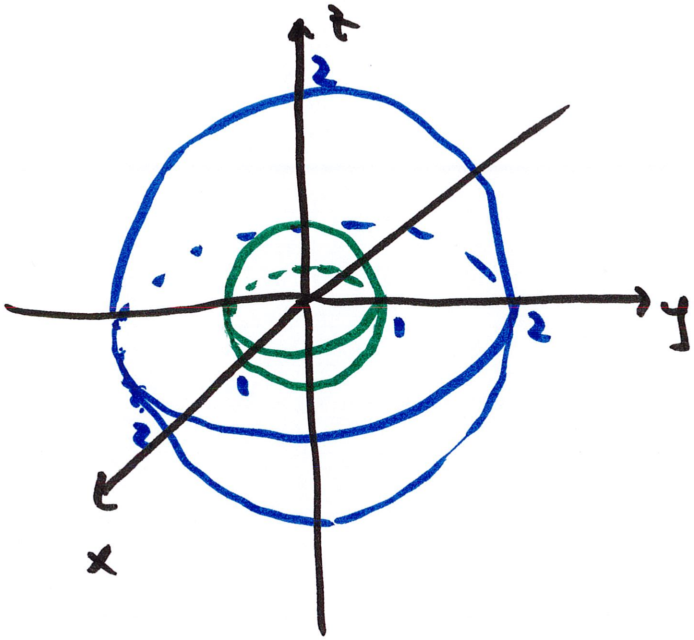
by Divergence Theorem again,
$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iiint_{D_1} \operatorname{div}\vec{F}\,dV$$
$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iiint_{D_2} \operatorname{div}\vec{F}\,dV$$

this then tells us:
$$\boxed{\iint_{S} \vec{F}\cdot\vec{n}\,dS = \iiint_{D_1} \operatorname{div}\vec{F}\,dV - \iiint_{D_2} \operatorname{div}\vec{F}\,dV}$$

the flux of a hollowed out volume is the difference of the volume integrals of the divergence example $\vec{F} = \langle x, y, z \rangle$

D : between spheres of radii 2 and 1

normal outward (away from enclosed volume)



we can do either

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV$$

↳ between spheres

or $$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_{D_1} \operatorname{div} \vec{F}\, dV - \iiint_{D_2} \operatorname{div} \vec{F}\, dV$$

D_1 ↳ inside big sphere

D_2 ↳ inside small sphere

let's try $\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_D \operatorname{div} \vec{F}\, dV$ first

D : $1 \le \rho \le 2$

$0 \le \phi \le \pi$

$0 \le \theta \le 2\pi$

$\vec{F} = \langle x, y, z \rangle \quad \text{div}\,\vec{F} = 3$

So the flux is

$$\int_0^{2\pi}\int_0^{\pi}\int_1^{2} 3\,\underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

$\text{div}\,\vec{F}$

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3 \Big|_{\rho=1}^{\rho=2} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 7 \sin\phi \, d\phi \, d\theta = 7\int_0^{2\pi} -\cos\phi \Big|_0^{\pi} d\theta = 7(2)\cdot 2\pi = \boxed{28\pi}$$

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let's try $\displaystyle\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_{D_1} \text{div}\,\vec{F}\, dV - \iiint_{D_2} \text{div}\,\vec{F}\, dV$

D_1: ↳ inside outer sphere

D_2: ↳ inside inner sphere

$$= \iiint_{D_1} 3\, dV - \iiint_{D_2} 3\, dV = 3\underbrace{\iiint_{D_1} dV}_{\text{volume of outer sphere}} - 3\underbrace{\iiint_{D_2} dV}_{\text{volume of inner sphere}}$$

$= 3 \cdot \frac{4}{3}\pi(2)^3 - 3\cdot\frac{4}{3}\pi(1)^3 = 32\pi - 4\pi = \boxed{28\pi}$

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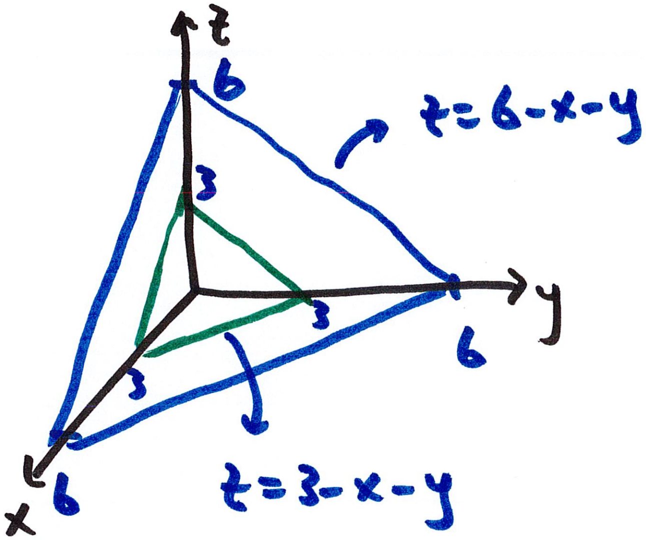
example $\vec{F} = \langle x^2, -y^2, z^2 \rangle$

D : bounded by $z = 6-x-y$ and $z = 3-x-y$ in first octant



D is space between the planes

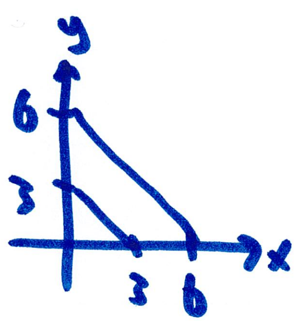
calculate flux

choices:

1) surface integral over 5 surfaces (big outside, small inside, 3 sides on coordinate planes)

2) $\iiint_D \operatorname{div}\vec{F}\,dV$

↳ space between planes



looks to be the easiest choice →

3) $\iiint_{D_1} \operatorname{div}\vec{F}\,dV - \iiint_{D_2} \operatorname{div}\vec{F}\,dV$

D_1 ↳ volume contained by the blue surface

D_2 ↳ volume contained by green surface

D_1: space contained by blue plane, ignoring the green one

$$0 \le x \le 6$$
$$0 \le y \le 6-x$$
$$0 \le z \le 6-x-y$$

D_2: space contained by the green plane

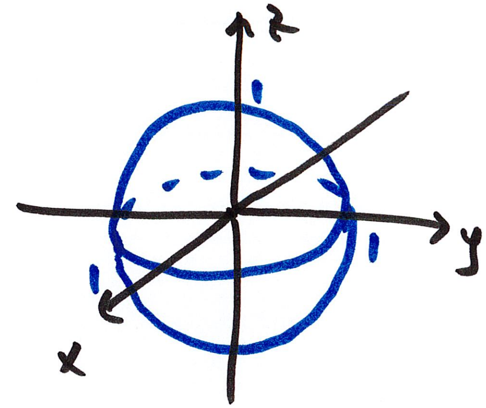
$$0 \le x \le 3$$
$$0 \le y \le 3-x$$
$$0 \le z \le 3-x-y$$

$$\operatorname{div} \vec{F} = 2x - 2y + 2z$$

$$\underbrace{\int_0^6 \int_0^{6-x} \int_0^{6-x-y} 2(x-y+z)\,dz\,dy\,dx}_{\iiint_{D_1} \operatorname{div} \vec{F}\, dV} - \underbrace{\int_0^3 \int_0^{3-x} \int_0^{3-x-y} 2(x-y+z)\,dz\,dy\,dx}_{\iiint_{D_2} \operatorname{div} \vec{F}\, dV}$$

$$= \cdots = \boxed{\frac{405}{4}}$$

example $\vec{F} = \dfrac{\langle x, y, z \rangle}{\sqrt{x^2+y^2+z^2}}$



S : sphere of radius 1

calculate flux

Divergence Theorem: $\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_D \operatorname{div} \vec{F} \, dV$

for this to work, $\operatorname{div} \vec{F}$ MUST be defined over D

$\operatorname{div} \vec{F} = \cdots = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$ is NOT defined everywhere in D

problem at $(0, 0, 0)$

The Divergence Theorem <u>CANNOT</u> be applied directly epsilon

there is a workaround: make up a tiny sphere radius ε enclosing the origin, then basically redo the sphere example from earlier and then take limit $\varepsilon \to 0$

$$\operatorname{div}\vec{F} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{\rho} \text{ in spherical}$$

space ~~between~~ spheres:

$$\varepsilon \le \rho \le 1$$
$$0 \le \phi \le \pi$$
$$0 \le \theta \le 2\pi$$

$$\iiint_D \operatorname{div}\vec{F}\, dV = \underset{\theta}{\int_0^{2\pi}} \underset{\phi}{\int_0^{\pi}} \underset{\rho}{\int_\varepsilon^{1}} \underset{\operatorname{div}\vec{F}}{\frac{2}{\rho}} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV} = \dots = 4\pi(1-\varepsilon^2)$$

now push ε to 0

$$\lim_{\varepsilon \to 0} 4\pi(1-\varepsilon^2) = \boxed{4\pi}$$