14.1 Vector-Valued Functions

Scalar-valued functions : $\underset{\text{scalar input}}{f(t)} = \underbrace{t^3 + 3}_{\text{scalar output}}$

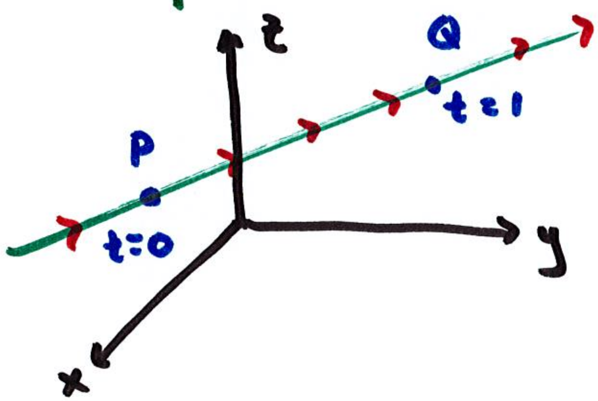
vector-valued functions : $\underset{\text{scalar input}}{\vec{r}(t)} = \underbrace{\langle \cos t, \sin t, 0 \rangle}_{\text{vector output}}$

we have seen vector-valued functions already

line : $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

line through $P(1,2,3)$ and $Q(4,5,6)$

$\underset{\text{scalar input}}{\vec{r}(t)} = \langle 1,2,3 \rangle + t\langle 3,3,3 \rangle = \underbrace{\langle 1+3t, 2+3t, 3+3t \rangle}_{\text{vector output}}$



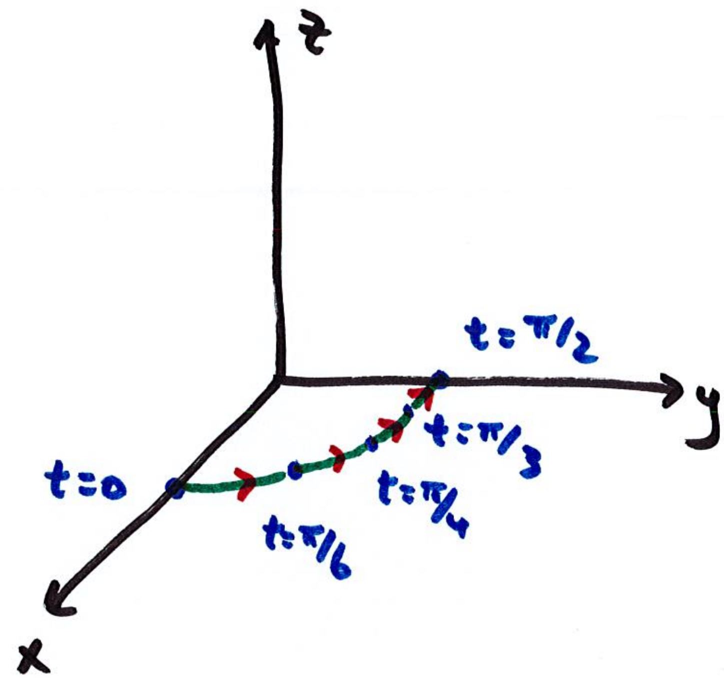
the direction of <u>increasing</u> t is called the <u>positive orientation</u>

example $\vec{r}(t) = \langle \underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{0}_{z} \rangle \qquad 0 \le t \le \pi/2$

graph?

one easy way to visualize: find points, connect them

t	x	y	z
0	1	0	0
$\pi/6$	$\sqrt{3}/2$	$1/2$	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	0
$\pi/3$	$1/2$	$\sqrt{3}/2$	0
$\pi/2$	0	1	0



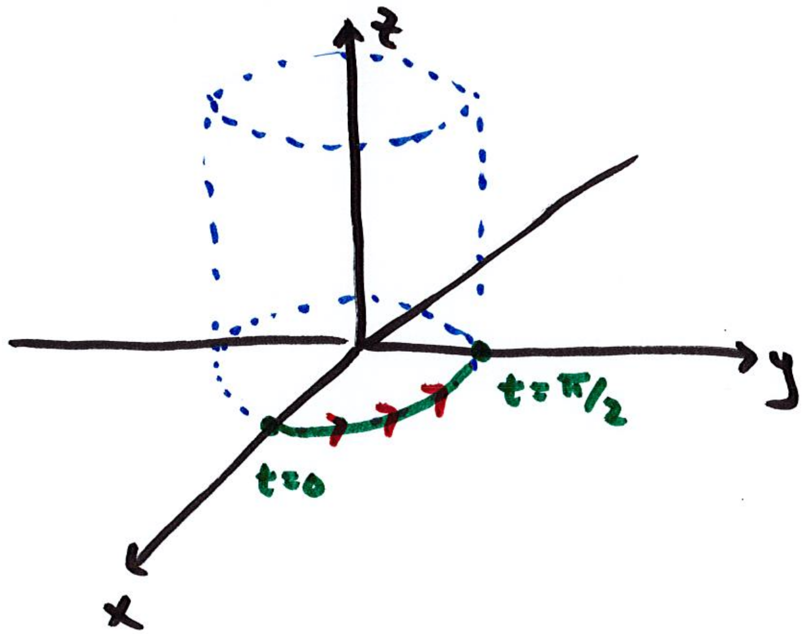
part of a circle moving counterclockwise when viewed from above another way to visualize : make a connection to a surface

$\vec{r}(t) = \langle \underbrace{\cos t}_{x}, \underbrace{\sin t}_{y}, \underbrace{0}_{z} \rangle \qquad 0 \le t \le \pi/2$

relationship between x, y, z ?

we know $\cos^2 t + \sin^2 t = 1 \longrightarrow \boxed{x^2 + y^2 = 1}$ in $\mathbb{R}^3$, this is a cylinder

the graph of $\vec{r}(t)$ is part of the surface



$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le \pi/2$

is a part of the surface

w/ $z = 0$

example

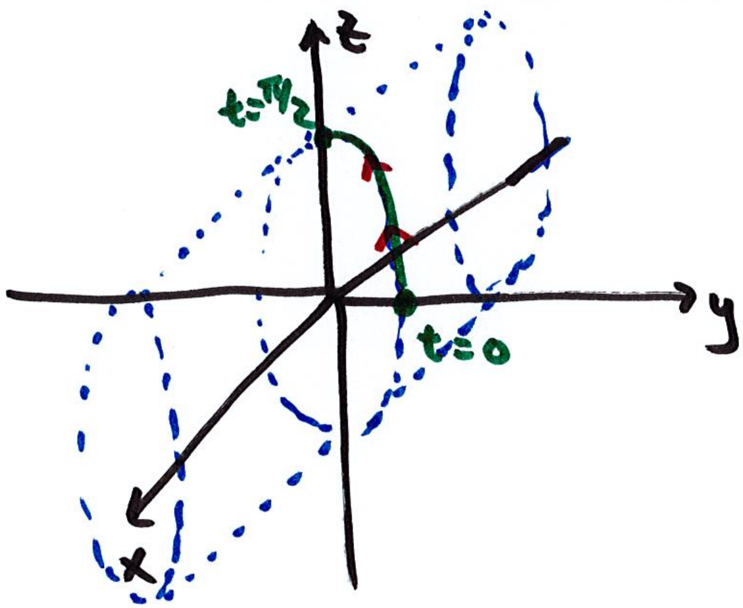
$$\vec{r}(t) = \langle \underset{x}{0},\ \underset{y}{\cos t},\ \underset{z}{2\sin t} \rangle \qquad 0 \le t \le \pi/2$$

$y = \cos t$, $z = 2\sin t$

we can see that

$$\begin{aligned}(2y)^2 + (z)^2 &= (2\cos t)^2 + (2\sin t)^2 \\ &= 4\cos^2 t + 4\sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) = 4\end{aligned}$$

now we see: $\boxed{4y^2 + z^2 = 4}$ in $\mathbb{R}^3$, this is a elliptic cylinder



$\vec{r}(t)$ starts at $t=0$

$\vec{r}(0) = \langle 0, 1, 0 \rangle$

$\vec{r}(\pi/2) = \langle 0, 0, 2 \rangle$

example

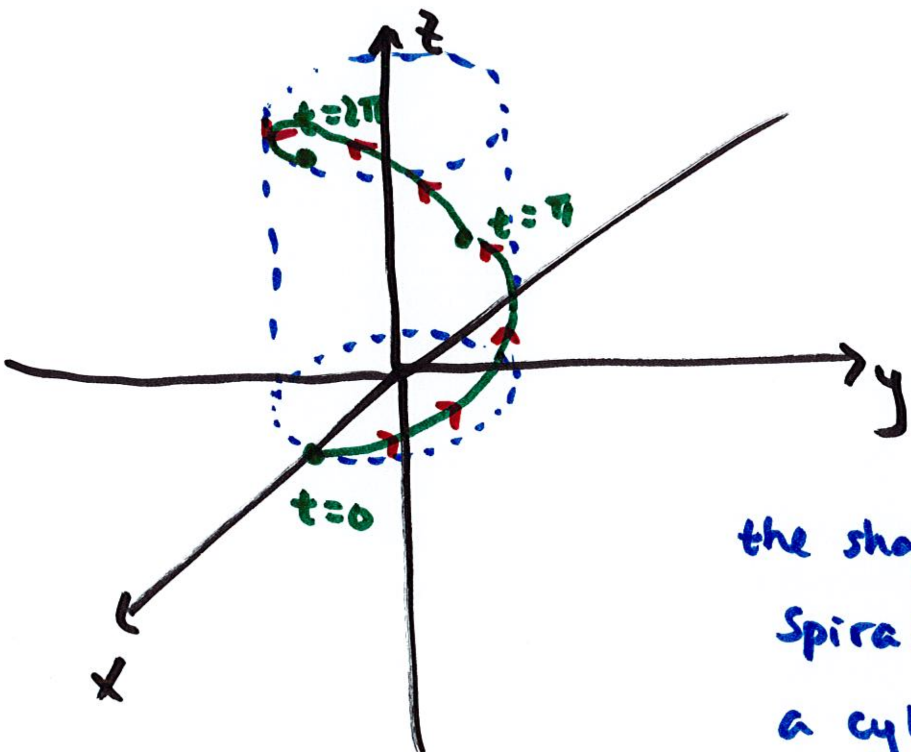
$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad 0 \le t \le 2\pi$

$\quad\quad\quad\quad x \quad\quad y \quad\quad z$

we know $\cos^2 t + \sin^2 t = 1$

so $x^2 + y^2 = 1 \rightarrow$ circular cylinder

$z = t$ tells us we are not just stuck on a particular slice of cylinder w/ constant z



$\vec{r}(0) = \langle 1, 0, 0 \rangle$

$\vec{r}(\pi) = \langle -1, 0, \pi \rangle$

$\vec{r}(2\pi) = \langle 1, 0, 2\pi \rangle$

the shape of $\vec{r}(t)$ is a spiral on the surface of a cylinder example Does the curve $\underbrace{\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle}$ $\quad 0 \le t \le 2\pi$

intersect the plane $x - z = 0$? → "space curves"

what does $\vec{r}(t) = \langle t\cos t,\ t,\ t\sin t \rangle$ look like?

on what surface?

$x = t\cos t$

$y = t$

$z = t\sin t$

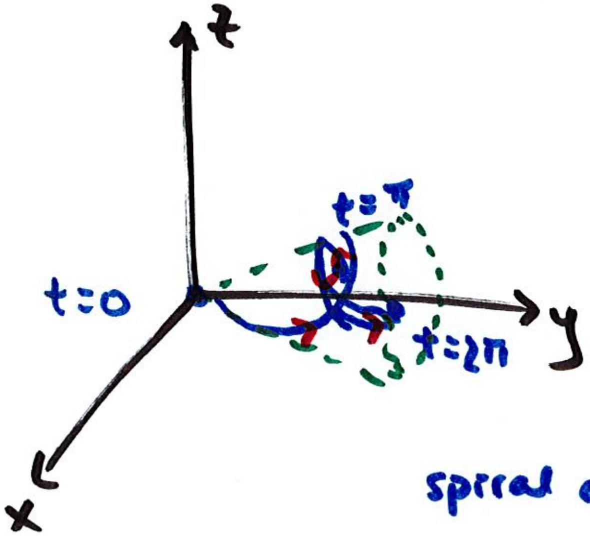
$$x^2 + z^2 = t^2\cos^2 t + t^2 \sin^2 t$$

$$= t^2(\cos^2 t + \sin^2 t)$$

$$x^2 + z^2 = t^2$$

↳ notice $y = t$

so this is $\boxed{x^2 + z^2 = y^2}$ cone



$t=0$: $\vec{r}(0) = \langle 0, 0, 0 \rangle$

$t=\pi$: $\vec{r}(\pi) = \langle -\pi, \pi, 0 \rangle$

$t=2\pi$: $\vec{r}(2\pi) = \langle 2\pi, 2\pi, 0 \rangle$

spiral on a cone

does $\vec{r}(t) = \langle \overset{x}{t\cos t}, \overset{y}{t}, \overset{z}{t\sin t} \rangle$ $\quad 0 \le t \le 2\pi$

intersect the plane $x - z = 0$?

intersection: point on $\vec{r}(t)$ is also on $x - z = 0$

point on $\vec{r}(t)$ <u>MUST</u> satisfy $x - z = 0$

so, $x = t\cos t$, $z = t\sin t$

solve $x - z = t\cos t - t\sin t = 0$

$$t(\cos t - \sin t) = 0 \rightarrow t = 0, \quad \underbrace{\cos t = \sin t}_{\text{between } 0 \le t \le 2\pi}$$

$t = \pi/4, \ 5\pi/4$

so curve $\vec{r}(t)$ intersects $x - z = 0$ 3 times

at $t = 0$, $t = \pi/4$, $t = 5\pi/4$

the domain of a vector-valued function is the intersection of the sub domains of all components

(in other words, where all components exist)

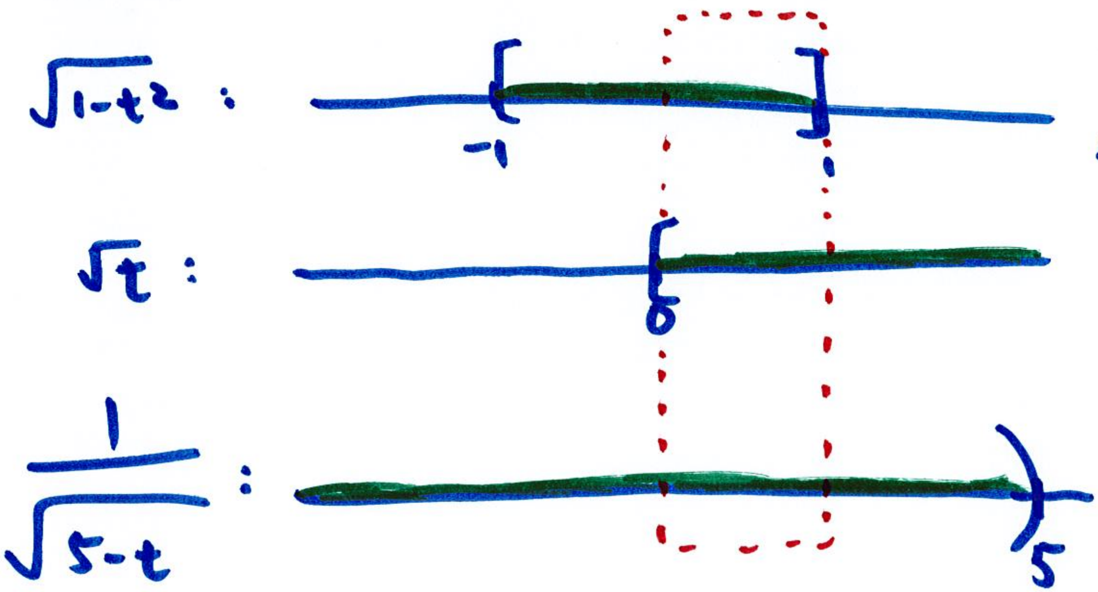
<u>example</u> $\vec{r}(t) = \left\langle \sqrt{1-t^2}, \sqrt{t}, \dfrac{1}{\sqrt{5-t}} \right\rangle$

$\sqrt{1-t^2}$ defined on $[-1, 1]$

$\sqrt{t}$ defined on $[0, \infty)$

$\dfrac{1}{\sqrt{5-t}}$ defined on $(-\infty, 5)$

the intersection is where all 3 parts are defined



overlap on $[0, 1]$

so domain of $\vec{r}(t) = \left\langle \sqrt{1-t^2}, \sqrt{t}, \dfrac{1}{\sqrt{5-t}} \right\rangle$ is $[0, 1]$