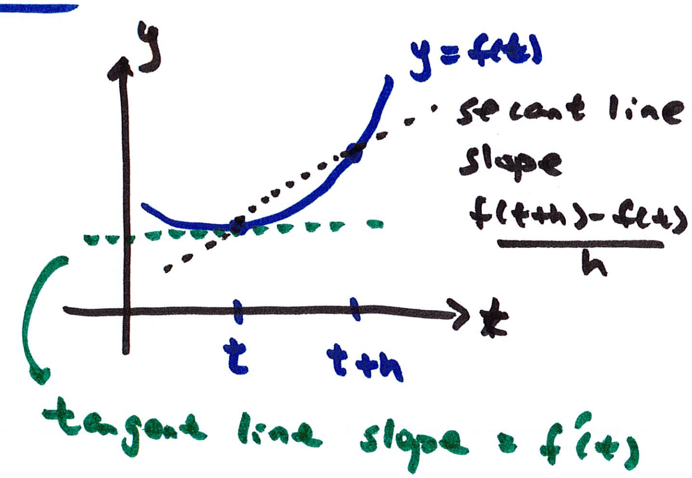
14.2 Calculus of Vector-Valued Functions

recall if $y = f(t)$ is a scalar function

then $y' = f'(t) = \lim_{h \to 0} \dfrac{f(t+h) - f(t)}{h}$

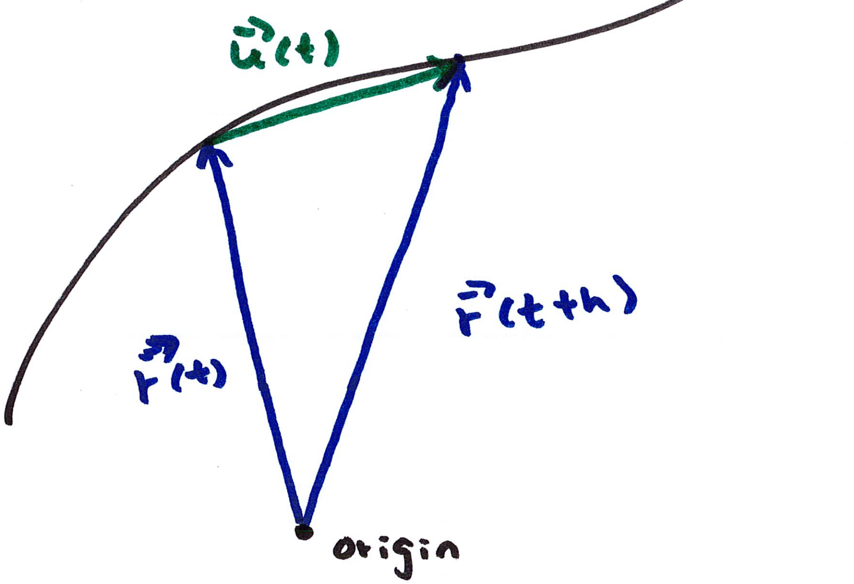


if $\vec{r}(t) = \langle x(t), y(t) \rangle$, the derivative is defined the same way

$$\boxed{\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}}$$

what is the geometric interpretation?

curve from $\vec{r}(t)$

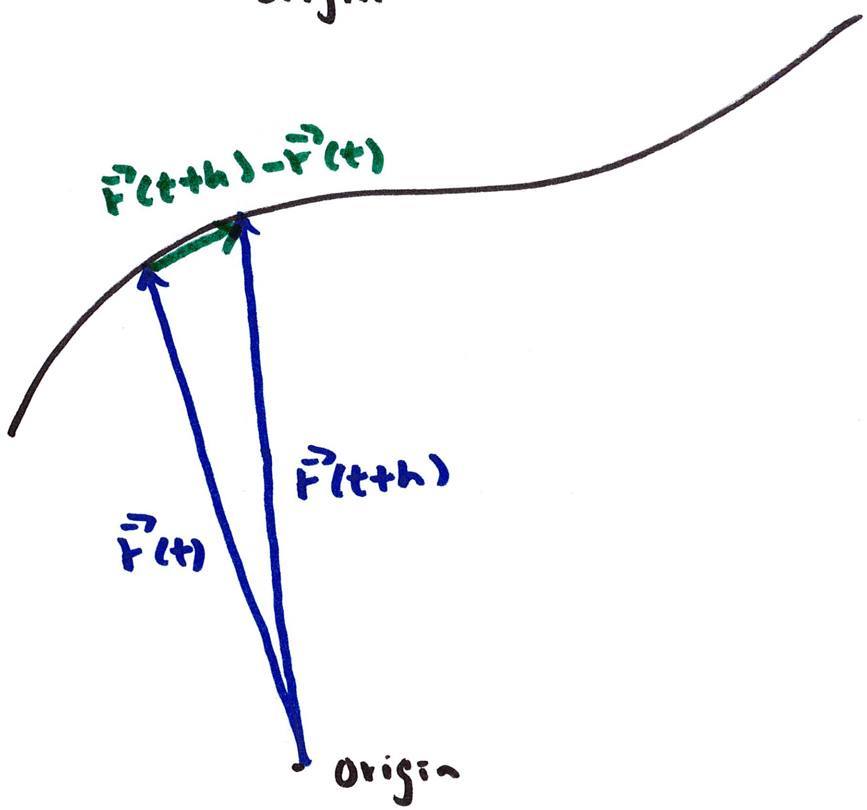


notice $\vec{r}(t) + \vec{u}(t) = \vec{r}(t+h)$

then $\vec{u}(t) = \vec{r}(t+h) - \vec{r}(t)$

numerator of definition of $\vec{r}\,'(t)$

$\vec{r}(t)$



as ~~t→~~ $h \to 0$

$\vec{r}(t+h) - \vec{r}(t)$ eventually becomes <u>tangent</u> to $\vec{r}(t)$ at t (same instantaneous direction)

as expected,

$\vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$ is the <u>tangent vector</u>

if $\vec{r}(t) = \langle x(t), y(t) \rangle$

then $$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\langle x(t+h), y(t+h) \rangle - \langle x(t), y(t) \rangle}{h}$$

$$= \lim_{h \to 0} \frac{\langle x(t+h) - x(t), y(t+h) - y(t) \rangle}{h}$$

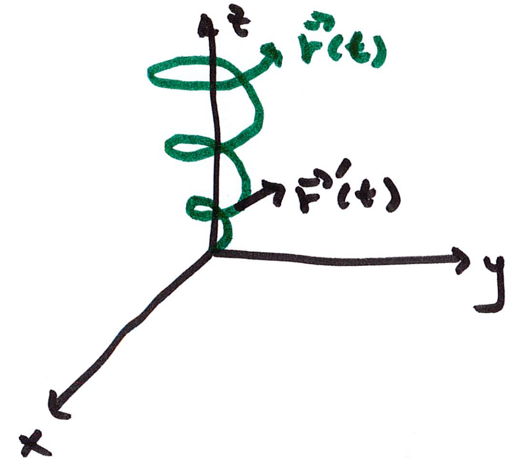
$$= \lim_{h \to 0} \left\langle \frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h} \right\rangle = \langle x'(t), y'(t) \rangle$$

same idea if $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

<u>example</u>

$\vec{r}(t) = \langle \underset{x}{t\cos t}, \underset{y}{t\sin t}, \underset{z}{t} \rangle$



$$x^2+y^2 = t^2\cos^2 t + t^2 \sin^2 t$$
$$= t^2(\cos^2 t + \sin^2 t) = t^2 = z^2$$

$x^2+y^2=z^2$ cone

$$\vec{r}\,'(t) = \langle -t\sin t + \cos t, \ t\cos t + \sin t, \ 1 \rangle$$

$\vec{r}\,'(\frac{\pi}{2}) = \langle -\frac{\pi}{2}, 1, 1 \rangle$ vector tangent to $\vec{r}(t)$ at $t = \pi/2$

$$|\vec{r}\,'(t)| = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2 + 1^2}$$
$$= \cdots\cdots = \sqrt{t^2+2}$$

magnitude changes as ~~ch~~ t changes

in general, $\vec{r}'(t)$ is NOT a unit vector

but some things later in the course need the ~~un~~ tangent vector as a unit vector

define <u>unit tangent vector</u>

$$\boxed{\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}}$$

for the spiral example,

$$\cancel{\vec{T} = \left\langle \frac{-t\sin t + \sin t}{\sqrt{t^2+2}}, \right.} \; t \, co$$

$$\vec{T} = \left\langle \frac{-t\sin t + \cos t}{\sqrt{t^2+2}}, \frac{t\cos t + \sin t}{\sqrt{t^2+2}}, \frac{1}{\sqrt{t^2+2}} \right\rangle$$

many things don't change, for example, the product rule (for dot product)

$$\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$= \frac{d}{dt}\left[\vec{v}(t)\cdot\vec{u}(t)\right]$$

order does NOT matter for dot product

$$\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \underbrace{\vec{u}'(t)\times\vec{v}(t)}_{\text{keep this order}} + \underbrace{\vec{u}(t)\times\vec{v}'(t)}_{\text{keep this order}}$$

order matters

integrals work mostly as expected

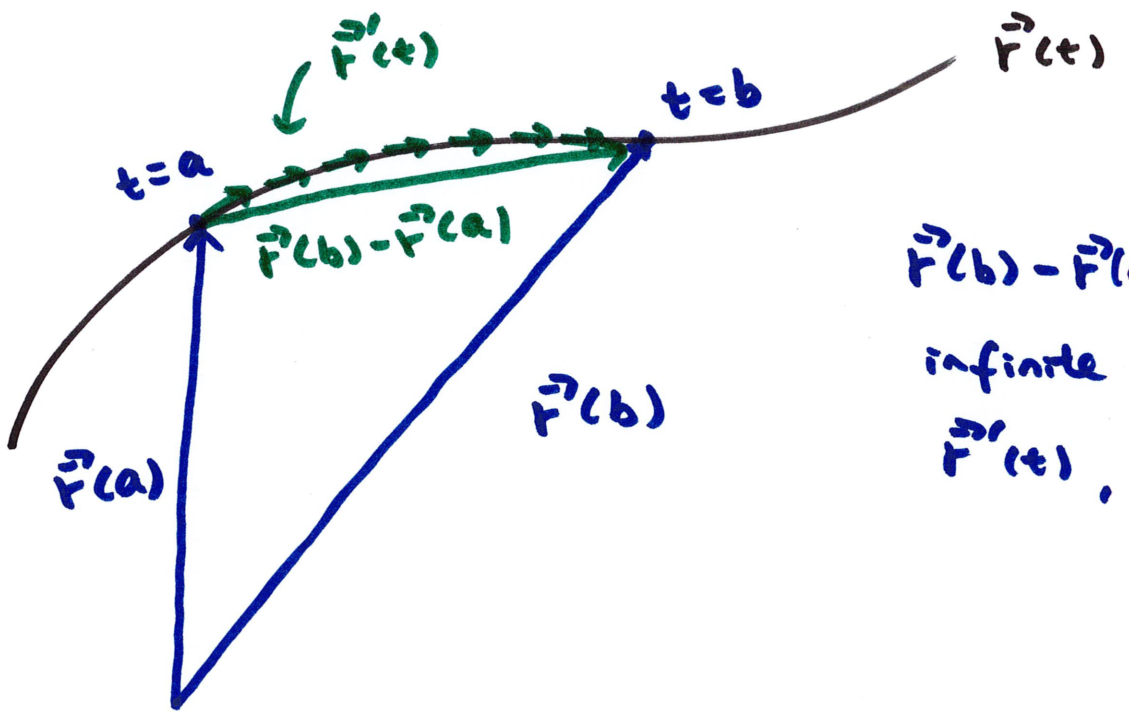
$$\int \vec{r}\,'(t)\,dt = \vec{r}(t) + \vec{C}$$

vector constant

$$\int_a^b \vec{r}\,'(t)\,dt = \underbrace{\vec{r}(b) - \vec{r}(a)}_{\text{vector}}$$

displacement vector from $t=a$ to $t=b$

geometric interpretation?



$\vec{r}(b) - \vec{r}(a)$ is the infinite sum of all $\vec{r}\,'(t)$, $a \le t \le b$

14.3 Motions in Space (part 1)

if $\vec{r}(t)$ describes the position / path of a object

then $\vec{r}\,'(t) = \vec{v}(t)$ is the <u>velocity vector</u>

$|\vec{r}\,'(t)|$ is the <u>speed</u>

and $\vec{r}\,''(t) = \vec{v}\,'(t) = \vec{a}(t)$ is the <u>acceleration vector</u>

<u>example</u> If $\vec{a}(t) = \langle 1, t, t^2 \rangle$ is the acceleration $(t \geq 0)$
find the position $\vec{r}(t)$ such that

$$\underbrace{\vec{v}(0) = \langle 1, 2, 3 \rangle}_{\text{initial velocity}} \quad \text{and} \quad \underbrace{\vec{r}(0) = \langle 0, 0, 0 \rangle}_{\text{initial position}}$$

$\vec{a}(t) = \langle 1, t, t^2 \rangle$

velocity: $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 1, t, t^2 \rangle\,dt$

$$= \left\langle \int 1\,dt,\ \int t\,dt,\ \int t^2\,dt \right\rangle$$

$$= \left\langle t + c_1,\ \tfrac{1}{2}t^2 + c_2,\ \tfrac{1}{3}t^3 + c_3 \right\rangle = \left\langle t, \tfrac{1}{2}t^2, \tfrac{1}{3}t^3 \right\rangle + \vec{C}$$

$\langle c_1, c_2, c_3 \rangle$

find $\vec{C}$ using $\vec{v}(0) = \langle 1, 2, 3 \rangle$ (given)

$\vec{v}(0) = \langle c_1, c_2, c_3 \rangle = \langle 1, 2, 3 \rangle$

so $\vec{v}(t) = \left\langle t+1, \tfrac{1}{2}t^2 + 2, \tfrac{1}{3}t^3 + 3 \right\rangle$

position: $\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \tfrac{1}{2}t^2 + t + d_1,\ \tfrac{1}{6}t^3 + 2t + d_2,\ \tfrac{1}{12}t^4 + 3t + d_3 \right\rangle$

$$= \left\langle \tfrac{1}{2}t^2 + t,\ \tfrac{1}{6}t^3 + 2t,\ \tfrac{1}{12}t^4 + 3t \right\rangle + \vec{D}$$

find $\vec{D}$ using $\vec{r}(0) = \langle 0, 0, 0 \rangle$ (given)

$\langle d_1, d_2, d_3 \rangle = \langle 0, 0, 0 \rangle$

so $\boxed{\vec{r}(t) = \left\langle \tfrac{1}{2}t^2 + t,\ \tfrac{1}{6}t^3 + 2t,\ \tfrac{1}{12}t^4 + 3t \right\rangle}$