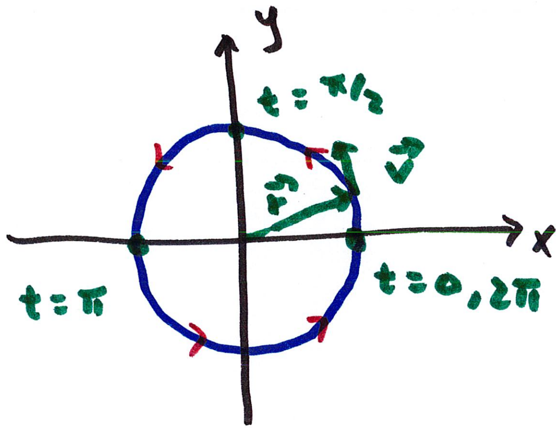
14.3 Motions in space (part 2)

circular motion: $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle \underset{x}{\cos t}, \underset{y}{\sin t} \rangle \quad 0 \le t \le 2\pi$

$x = \cos t, \quad y = \sin t$

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \rightarrow x^2 + y^2 = 1$



velocity: $\vec{r}'(t) = \vec{v}(t) = \langle -\sin t, \cos t \rangle$

~~we~~ see $\vec{v}(t) \cdot \vec{r}(t) = \langle -\sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle$

$= -\sin t \cos t + \cos t \sin t = 0$

so, $\vec{r}(t) \perp \vec{v}(t)$

speed: $|\vec{v}| = |\langle -\sin t, \cos t \rangle|$

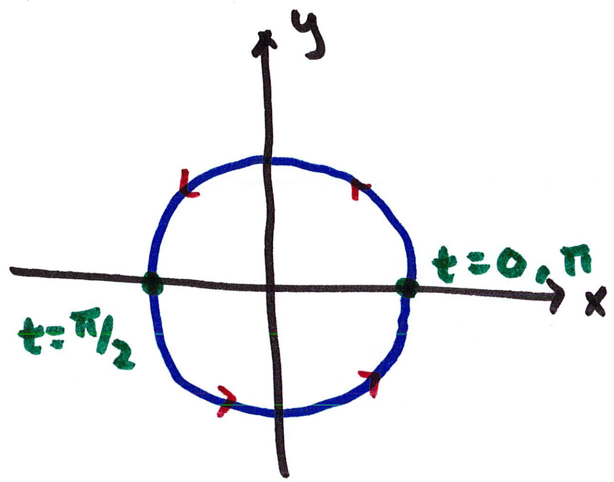
$= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

speed is constant but velocity is NOT

acceleration: $\vec{v}'(t) = \vec{a}(t) = \langle -\cos t, -\sin t \rangle$

note $\vec{r}(t)$ and $\vec{a}(t)$ are in opposite directions another circle: $\vec{r}(t) = \langle \cos(2t), \sin(2t) \rangle \qquad 0 \le t \le \pi$

still circle, since $x^2 + y^2 = \cos^2(2t) + \sin^2(2t) = 1$
same



this trajectory has the same points as the previous one, but completes in half the "time"

velocity: $\vec{v}(t) = \vec{r}\,'(t) = \langle -2\sin(2t), 2\cos(2t) \rangle$

speed: $|\vec{v}(t)| = \sqrt{4\sin^2(2t) + 4\cos^2(2t)}$

$$= \sqrt{4} = 2$$

higher speed makes sense because trajectory completed faster

$\vec{r}(t) = \langle R\cos t, R\sin t \rangle \rightarrow$ circle radius R

$\vec{r}(t) = \langle 2\cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

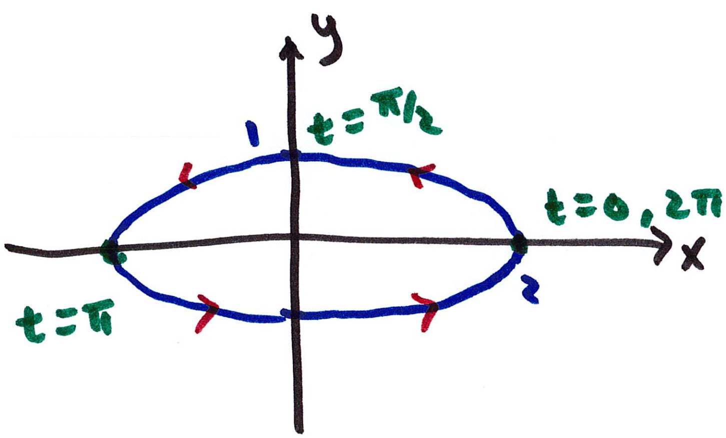
note: $x^2 + y^2 = 4\cos^2 t + \sin^2 t \neq 1$ or constant, so not a circle

but $x^2 + (2y)^2 = 4\cos^2 t + (2\sin t)^2$

$= 4\cos^2 t + 4\sin^2 t = 4$

$x^2 + 4y^2 = 4$

$\left(\frac{x}{2}\right)^2 + y^2 = 1$ ellipse



velocity: $\vec{v}(t) = \vec{r}\,' = \langle -2\sin t, \cos t \rangle$

$\vec{v} \cdot \vec{r} = \langle -2\sin t, \cos t \rangle \cdot \langle 2\cos t, \sin t \rangle$

$= -4\cos t \sin t + \cos t \sin t \neq 0$

so $\vec{r}$ and $\vec{v}$ are NOT orthogonal in general

acceleration: $\vec{a} = \vec{v}\,' = \langle -2\cos t, -\sin t \rangle$

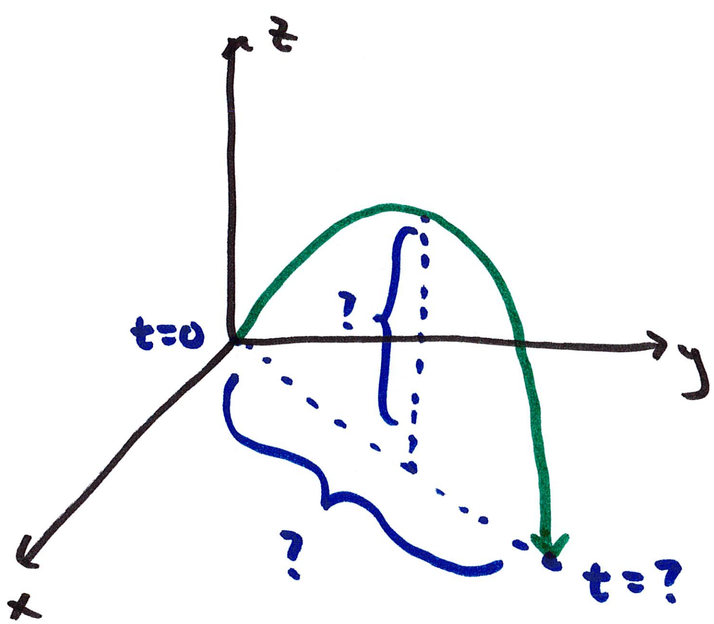
so $\vec{a}$ and $\vec{r}$ still in opposite direction now let's find the trajectory ($\vec{r}(t)$) given an acceleration

<u>example</u> A ball resting on the ground is launched into the air with the initial velocity $\vec{v}(0) = \langle 1.225, 2.45, 4.9 \rangle$ m/s. If gravity is the only force acting on it

find: a) How long ~~the~~ does the ball stay in the air?

b) How far did it travel?

c) How high did it go?



let's assume we start at the origin : $\vec{r}(0) = \langle 0, 0, 0 \rangle$

gravity acceleration : $\vec{a}(t) = \langle 0, 0, -9.8 \rangle$ m/s²

velocity : $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, 0, -9.8 \rangle\,dt$

$$\vec{v}(t) = \langle c_1, c_2, -9.8t + c_3 \rangle$$

set this (at $t=0$) equal to $\vec{v}(0) = \langle 1.225, 2.45, 4.9 \rangle$ to find c's

$$\vec{v}(0) = \langle c_1, c_2, c_3 \rangle = \langle 1.225, 2.45, 4.9 \rangle$$

so, $\boxed{\vec{v}(t) = \langle 1.225, 2.45, -9.8t + 4.9 \rangle}$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 1.225t + d_1, 2.45t + d_2, -4.9t^2 + 4.9t + d_3 \rangle$$

$$\vec{r}(0) = \langle d_1, d_2, d_3 \rangle = \langle 0, 0, 0 \rangle$$

so, $\boxed{\vec{r}(t) = \langle 1.225t, 2.45t, -4.9t^2 + 4.9t \rangle}$

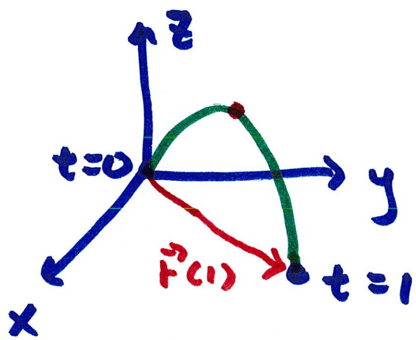
how much time in air? (time of flight?)

find t such that $z(t) = -4.9t^2 + 4.9t = 0$

$$4.9t(-t+1) = 0 \rightarrow t = 0, \ \boxed{t = 1}$$

time of flight

how far did it go? (range?)



$$\text{range} = |\vec{r}(1)| = |\langle 1.225, 2.45, 0 \rangle|$$
$$= \boxed{2.74}$$

max height? at peak, the z-component of velocity is 0

z velocity: $-9.8t + 4.9 = 0 \rightarrow \boxed{t = 1/2}$ max height at $t = 1/2$

the max height: $\underset{z \text{ at } t=1/2}{\cancel{|\vec{r}(1/2)|}} = 1.225$

what if we <u>doubled</u> the initial velocity?

does range double? does the time of flight double?

now $\vec{v}(0) = \langle 2.45, 4.9, 9.8 \rangle$ (twice of old $\vec{v}(0)$)

$$\left.\begin{aligned} \vec{a}(t) &= \langle 0, 0, -9.8 \rangle \\ \vec{r}(0) &= \langle 0, 0, 0 \rangle \end{aligned}\right\} \text{same}$$

follow same steps

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \cdots = \boxed{\langle 2.45, 4.9, -9.8t + 9.8 \rangle}$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \cdots = \boxed{\langle 2.45t, 4.9t, -4.9t^2 + 9.8t \rangle}$$

time of flight : $-4.9t^2 + 9.8t = 0 \rightarrow t = 0, \boxed{t = 2}$ double time of flight

range : $|\vec{r}(2)| = \cdots = 10.96 \rightarrow$ 4 times the old range