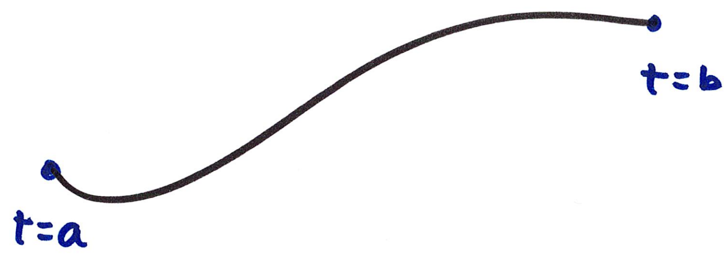
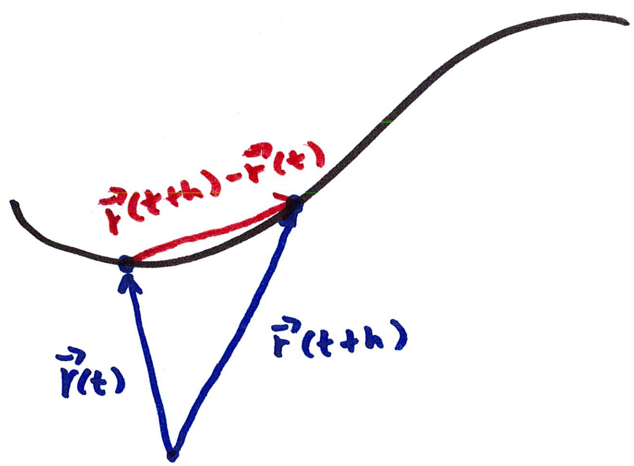
14.4 Length of Curves

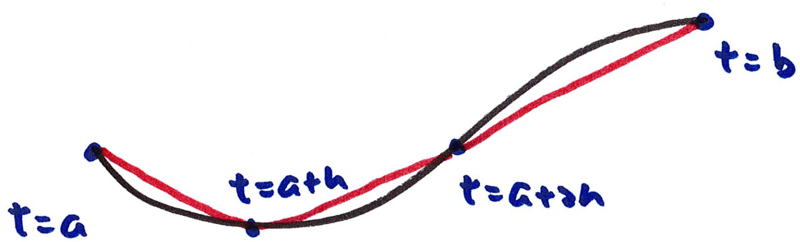
$\vec{r}(t)$ $\quad a \le t \le b$



how to find length from $t=a$ to $t=b$?



if h is small, then $|\vec{r}(t+h)-\vec{r}(t)| \approx$ true length during that interval t from t to $t+h$



we can approximate the true length by summing up the small straight segments from last time: $\vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

$$\vec{r}\,'(t) \approx \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$ when h is small

so, $\vec{r}(t+h) - \vec{r}(t) \approx \vec{r}\,'(t)h$

$$\underbrace{|\vec{r}(t+h) - \vec{r}(t)| \approx |\vec{r}\,'(t)|\,h}_{\text{the approximate length from } t \text{ to } t+h}$$

the total length of $\vec{r}(t)$ is therefore, approximately

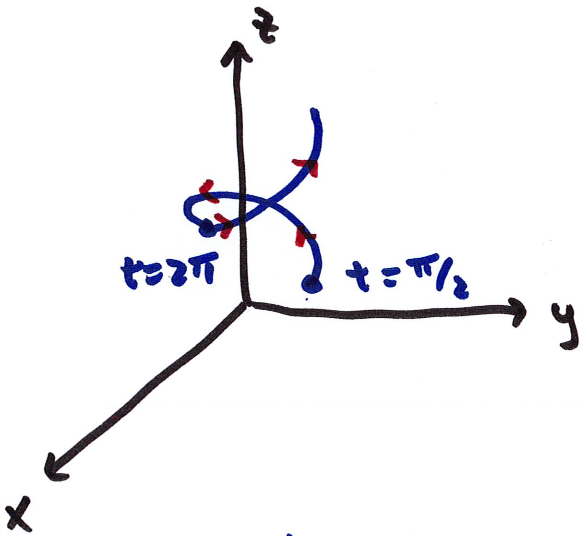
$$L \approx \sum_{i=1}^{n} |\vec{r}\,'(t_i)|\,\underbrace{h}_{\Delta t}$$ summing many small straight segments (n of them)

$$\approx \sum_{i=1}^{n} |\vec{r}\,'(t_i)|\,\Delta t$$

now we shrink Δt to $\Delta t \to dt$, $n \to \infty$

so, $L = \lim_{n \to \infty} \sum_{i=1}^{n} |\vec{r}\,'(t_i)|\,\Delta t = \boxed{\int_a^b |\vec{r}\,'(t)|\,dt}$ gives us the <u>exact</u> length of $\vec{r}(t)$ $a \le t \le b$

<u>example</u> $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad \frac{\pi}{2} \le t \le 2\pi$



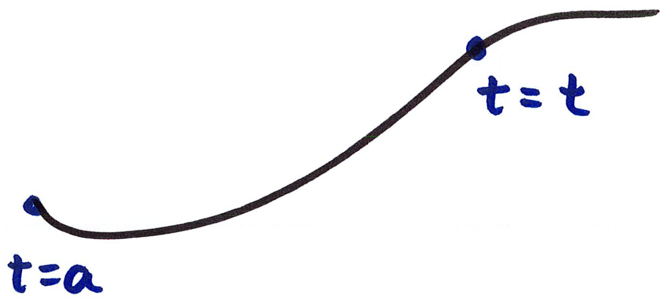
$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$= \int_{\pi/2}^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,t \Big|_{\pi/2}^{2\pi} = \sqrt{2}\left(2\pi - \pi/2\right) = \boxed{\frac{3\pi}{2}\sqrt{2}}$$

we can tweak $L = \int_a^b |\vec{r}\,'(t)|\,dt$ a bit to find a function that represents the length as a function of t



don't specify b, leave it as t

$$L(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

u is called a dummy variable

we don't want to integrate in terms of the "live" variable t

so, the length function is

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du}$$

try it on the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $a = \pi/2$

$|\vec{r}\,'(t)| = \sqrt{2}$ from last example

$$s(t) = \int_{\pi/2}^t |\vec{r}\,'(u)|\,du = \int_{\pi/2}^t \sqrt{2}\,du = \boxed{\sqrt{2}(t - \pi/2)}$$

the length from $t = \pi/2$ to some t

same result as previous if $t = 2\pi$

$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$ tells us how $\underset{\text{length}}{s}$ and $\underset{\text{time}}{t}$ are related

back to $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ tells us where we are <u>at a given t</u>

if we change the parameter to s (= length), then

~~$\vec{r}(s)$~~ $\vec{r}(s)$ tells us where we are after having <u>traveled a given s</u>

to find $\vec{r}(s)$, we need $s(t)$

back to $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ $\quad t \geq \pi/2$

from earlier, $|\vec{r}\,'(t)| = \sqrt{2}$ and $s(t) = \int_{\pi/2}^{t} \sqrt{2}\,du = \sqrt{2}(t - \pi/2)$

$s = \sqrt{2}(t - \pi/2) \rightarrow \underbrace{t = \frac{s}{\sqrt{2}} + \frac{\pi}{2}}_{\text{replace } t \text{ in } \vec{r}(t)}$

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \right\rangle}$$

this gives us location after having traveled a length s

for example, $\vec{r}(1) = \left\langle \cos\left(\frac{1}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{1}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{1}{\sqrt{2}} + \frac{\pi}{2} \right\rangle$ where we are after traveling dist of 1

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ $\qquad \frac{\pi}{2} \le t \le 2\pi$ $\qquad s = \sqrt{2}\left(t - \frac{\pi}{2}\right)$

$\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \right\rangle$ $\qquad 0 \le s \le \frac{3\pi}{\sqrt{2}}$

s when $t = \pi/2$ $\qquad$ s when $t = 2\pi$

back to $\quad s(t) = \displaystyle\int_a^t |\vec{r}\,'(u)|\,du$

$$\frac{ds}{dt} = \underbrace{\frac{d}{dt}\int_a^t |\vec{r}\,'(u)|\,du = |\vec{r}\,'(t)|}_{\text{Fundamental Theorem of Calculus}}$$

notice if $\frac{ds}{dt} = 1 = |\vec{r}\,'(t)|$

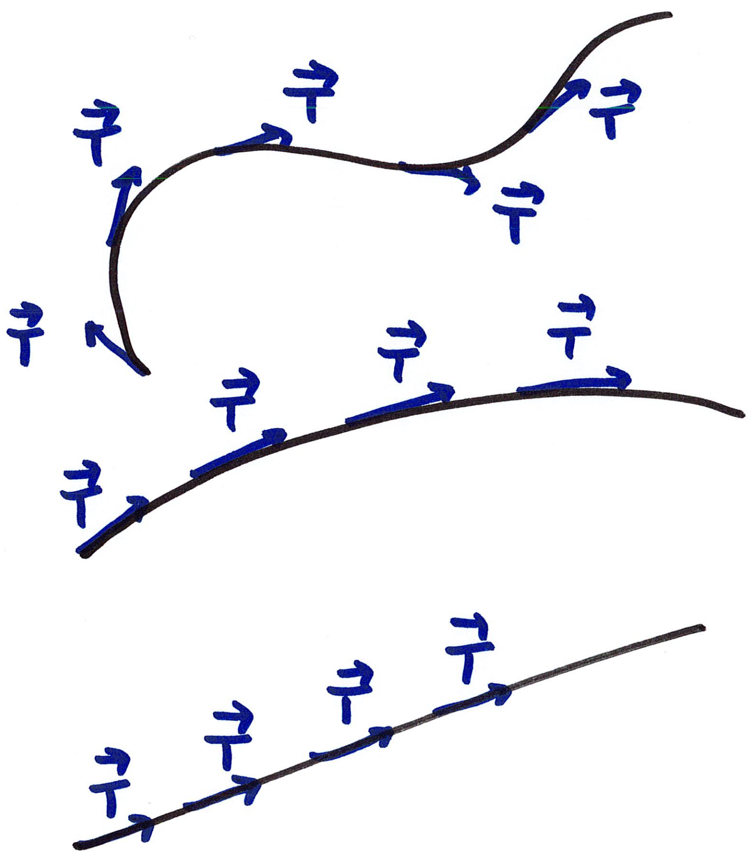
this means s and t increase at the same rate

so, in that case, the "s" in $\vec{r}(s)$ could actually also by "t"

14.5 Curvature

recall the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$\vec{T}$ cannot change length, so whatever change in $\vec{T}$ must be due to direction change



lots of turns, so we expect $\left|\dfrac{d\vec{T}}{ds}\right|$ to be big because of sharp turns

length

we expect smaller $\left|\dfrac{d\vec{T}}{ds}\right|$

straight line: $\left|\dfrac{d\vec{T}}{ds}\right| = 0$

we call $\left|\frac{d\vec{T}}{ds}\right|$ the <u>curvature</u> and its symbol is κ

Greek "kappa"

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

in practice, $\vec{T}$ is usually function of t, so taking derivative with respect to s is not always convenient

use chain rule to tweak it

from $\frac{ds}{dt} = |\vec{r}\,'(t)|$ we can rewrite $\left|\frac{d\vec{T}}{ds}\right|$ as

$$\kappa = \frac{|d\vec{T}|}{|ds|} = \frac{|d\vec{T}/dt|}{|ds/dt|} = \boxed{\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \kappa}$$

other formulas for κ: $\kappa = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3}$ or if $\vec{r}\,' = \vec{v}$, $\vec{r}\,'' = \vec{a}$

$$\kappa = \frac{|\vec{a} \times \vec{v}|}{|\vec{v}|^3}$$