15.1 Functions of Several Variables

we are familiar with one-variable functions

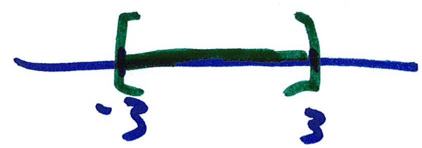
$$y = f(x) = \underbrace{\sqrt{9 - x^2}}_{\text{output}}$$

input

the set of all possible input values is called the <u>domain</u>

for this function, the domain is $9 - x^2 \geq 0$

$-3 \leq x \leq 3$ or $[-3, 3]$



note this is a line or sections of lines

the set of all possible output values is called the <u>range</u>

here, the range is $[\underbrace{0}_{\text{when } x = \pm 3}, \underbrace{3}_{\text{when } x = 0}]$

now let's look at a function of two variables

for example, $\underbrace{z = f(x, y)}_{\text{input}} = \underbrace{\sqrt{9-x^2} - \sqrt{25-y^2}}_{\text{output}}$

the input is now a set of ordered pairs (x, y)

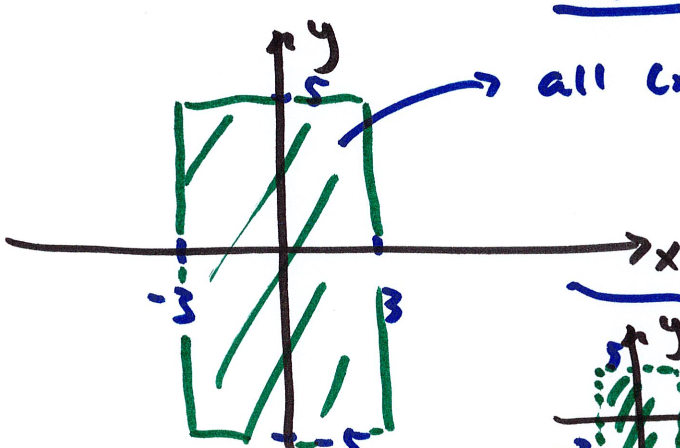
the output remains a scalar, like the one-variable case

the domain is now represented differently from 1-var case

here, we require $\quad 9 - x^2 \geq 0$ **AND** $25 - y^2 \geq 0$

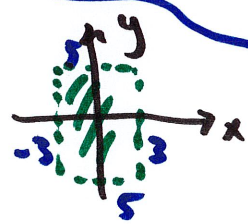
$\qquad -3 \leq x \leq 3$ **AND** $-5 \leq y \leq 5$

the domain is now a region on xy-plane



all (x, y) such that

if the end points are not included (( ) instead of [ ])



sometimes expressed as

$$\boxed{\{(x, y) : -3 \leq x \leq 3,\ -5 \leq y \leq 5\}}$$

the ~~range~~ output is still a single number, so the range is analyzed the same way

$$z = f(x,y) = \underbrace{\sqrt{9-x^2}}_{\substack{\text{largest}: 3 \\ \text{Smallest}: 0}} - \underbrace{\sqrt{25-y^2}}_{\substack{\text{largest}: 5 \\ \text{Smallest}: 0}}$$

so range is $-5 \le z \le 3$

or $[-5, 3]$

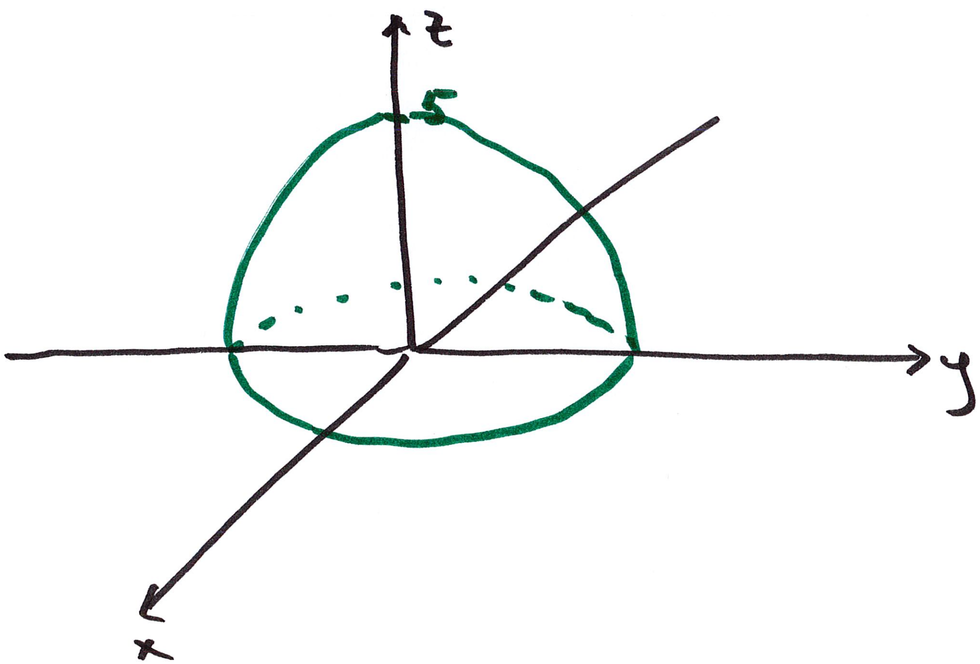
or $\{ z : -5 \le z \le 3 \}$

we know $z=f(x,y)$ gives us a <u>surface</u> (plane, sphere, etc)

if we set $z=z_0$ = constant, the the resulting graph $z_0=f(x,y)$ is called a <u>level curve</u> or a <u>contour</u> of $f(x,y)=z_0$

(essentially the same as a trace)

<u>example</u> $z=f(x,y)=5-x^2-y^2=5-(x^2+y^2)$ paraboloid



domain: $-\infty<x<\infty$

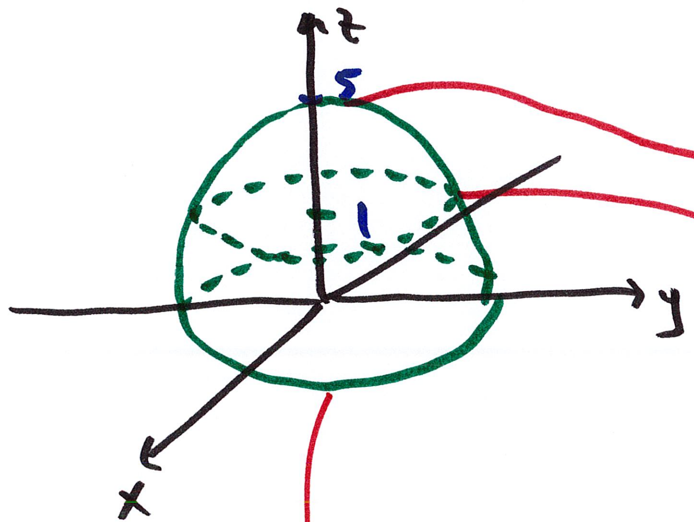
$-\infty<y<\infty$

$\{(x,y):\mathbb{R}^2\}$

range: $(-\infty, 5]$

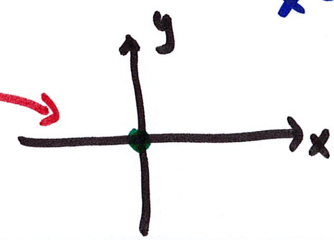
$z = f(x,y) = 5 - x^2 - y^2$

level curves : $z =$ constant



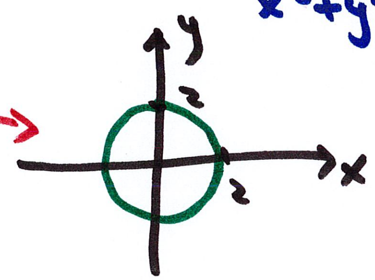
$z = z_0 = 5$: $5 = 5 - x^2 - y^2$

$x^2 + y^2 = 0$



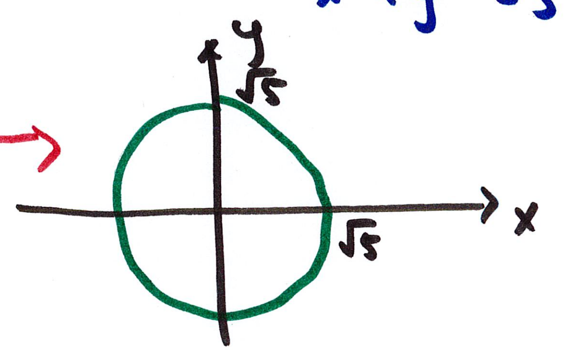
$z = z_0 = 1$: $1 = 5 - x^2 - y^2$

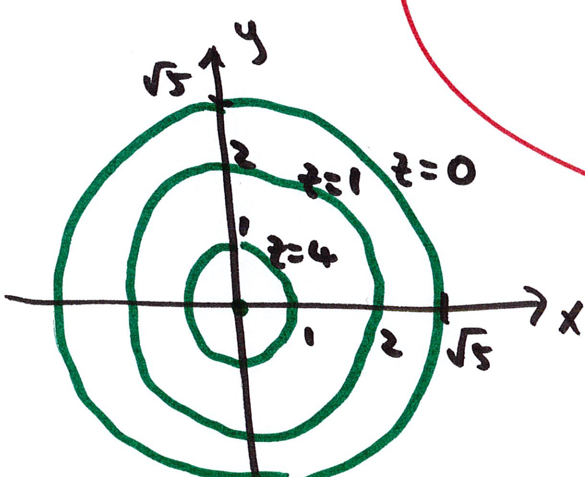
$x^2 + y^2 = 4$



$z = z_0 = 0$: $0 = 5 - x^2 - y^2$

$x^2 + y^2 = 5$





example $f(x,y) = \sin(xy)$

use the level curves to construct the surface

domain: $\{(x,y) : \mathbb{R}^2\}$ sine accepts any real number

range: $\{z : -1 \leq z \leq 1\}$ $-1 \leq \sin(\square) \leq 1$

level curves: $z = z_0$ = constant

$$z = f(x,y) = \underbrace{\text{constant} = \sin(xy)}$$

$$xy = \underbrace{\sin^{-1}(\text{constant})}_{\text{constant} = K}$$

so, level curves are $xy = K$ hyperbola

$\hookrightarrow y = \frac{K}{x}$ (x, y axes are asymptotes)

$y = \frac{k}{x}$ $\quad z_0 = \sin(k)$

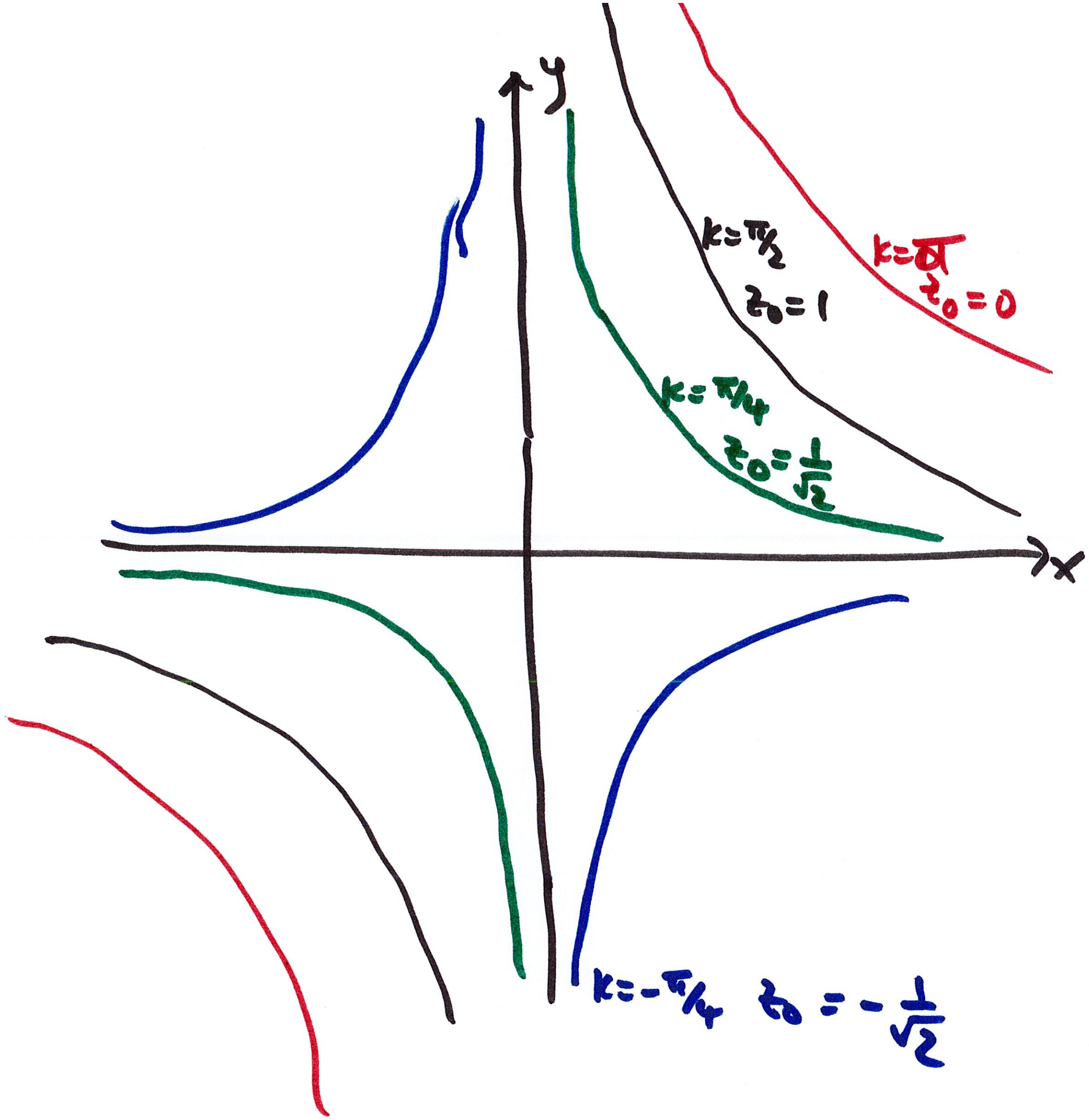
if $k = \frac{\pi}{4}$ $\quad z_0 = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

slice at $z = \frac{1}{\sqrt{2}}$

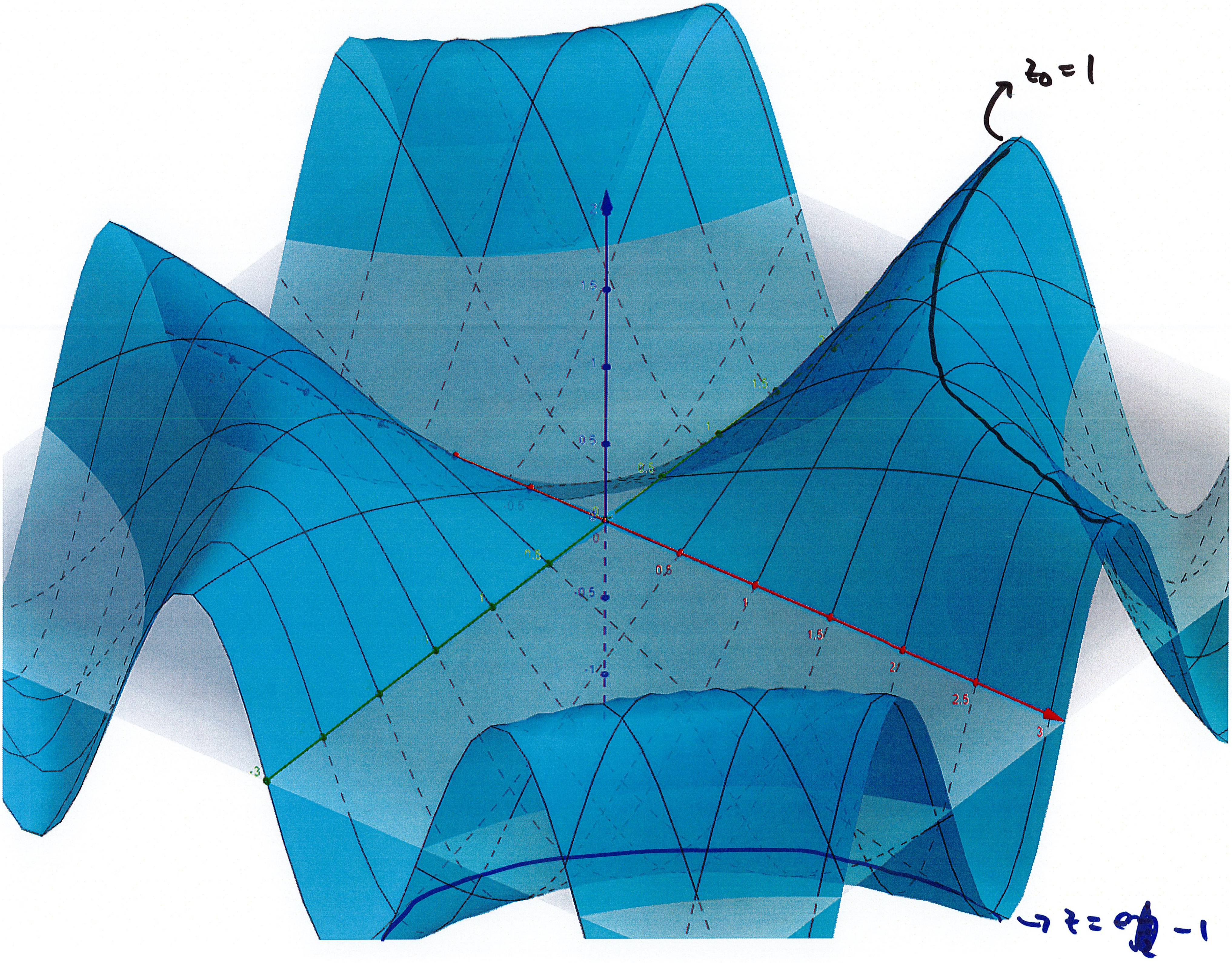
if $k = \frac{\pi}{2}$ $\quad z_0 = 1$

if $k = \pi$ $\quad z_0 = 0$

if $k = -\frac{\pi}{4}$ $\quad z_0 = -\frac{1}{\sqrt{2}}$

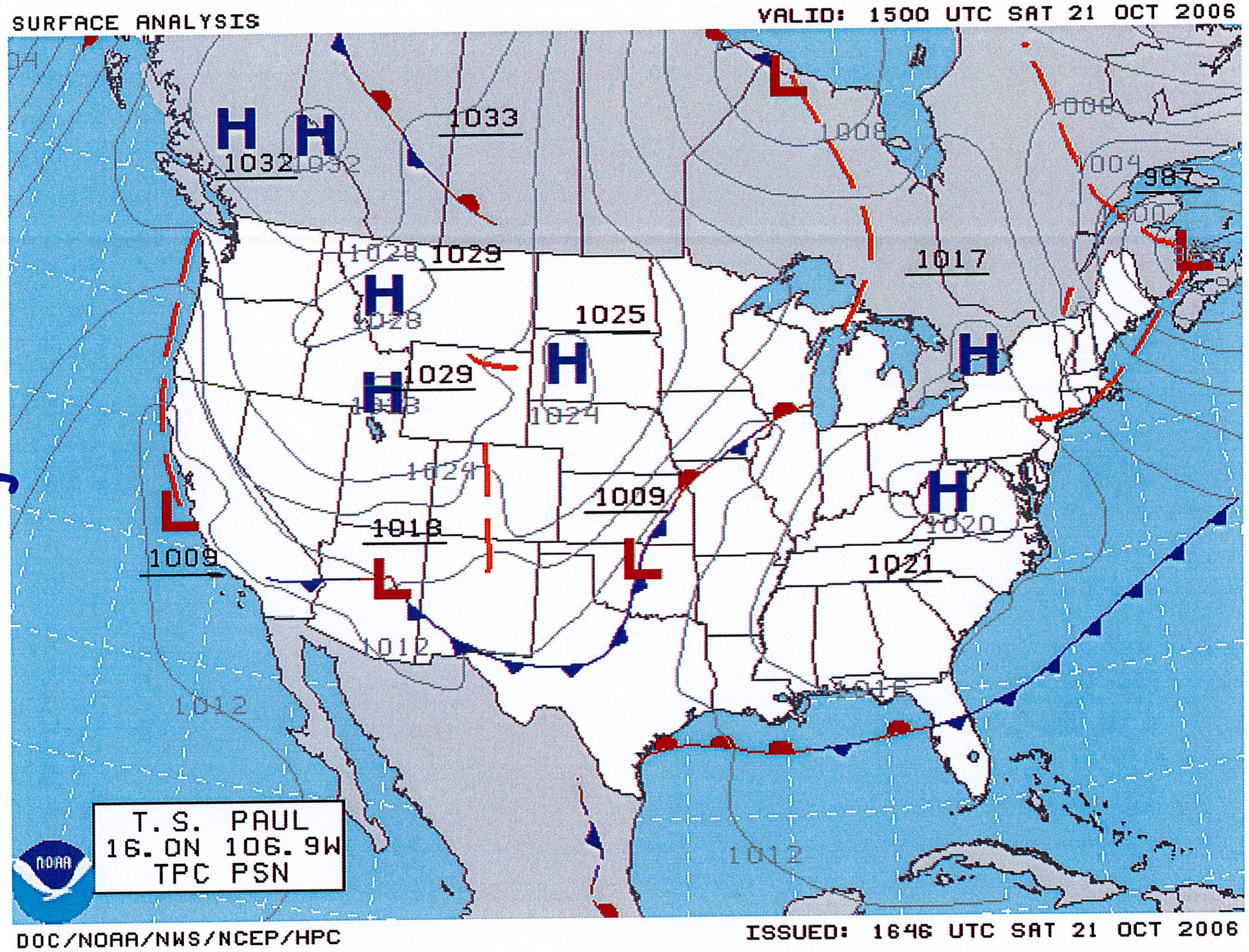


these are hyperbolic waves



z₀=1
z = -1

example of level curves



curves of constant pressure

isobaric curves