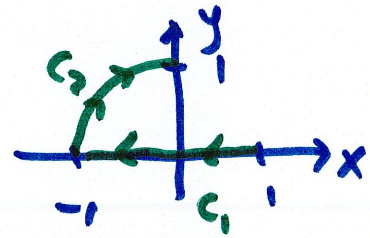
Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = \langle x + y, x\rangle$ and C is the line segment from (1, 0) to (− 1, 0) then along $y = 1 - x^2$ to (0, 1)

− 0.5

two ways: treat as line integral



$$C_1: \vec{r}(t) = \langle 1, 0\rangle + t\langle -2, 0\rangle$$

$$= \langle 1-2t, 0\rangle \quad 0 \le t \le 1 \rightarrow d\vec{r} = \vec{r}\,' dt = \langle -2, 0\rangle dt$$

$$C_2: \vec{r}(t) = \langle t, 1-t^2\rangle \quad -1 \le t \le 0 \rightarrow d\vec{r} = \langle 1, -2t\rangle dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \langle 1-2t, 1-2t\rangle \cdot \langle -2, 0\rangle dt + \int_{-1}^0 \langle 1-t^2+t, t\rangle \cdot \langle 1, -2t\rangle dt$$

$$= \int_0^1 -2+4t \, dt + \int_{-1}^0 1-t^2+t-2t^2 \, dt = \cdots = -\frac{1}{2}$$

the other way: notice $\vec{F} = \langle x+y, x\rangle$ is conservative

because $\frac{\partial}{\partial x}(x) = \frac{\partial}{\partial y}(x+y) = 1$

so, $\int_C \vec{F}\cdot d\vec{r} = \phi(\text{end}) - \phi(\text{start}) \qquad \vec{F} = \vec{\nabla}\phi$

Find ϕ: $\vec{F} = \vec{\nabla}\phi = \langle \phi_x, \phi_y \rangle$

$\langle x+y, x \rangle = \langle \phi_x, \phi_y \rangle$

$\phi_x = x+y$

$\phi_y = x$

$\phi = \int x+y \, dx = \frac{1}{2}x^2 + xy + h(y)$

$\phi_y = x + h'(y)$ must match

$= x$ $\qquad h'(y) = 0 \rightarrow h(y) = c$

so, $\phi = \frac{1}{2}x^2 + xy + c$

$\int_C \vec{F} \cdot d\vec{r} = \phi(\text{end}) - \phi(\text{start})$

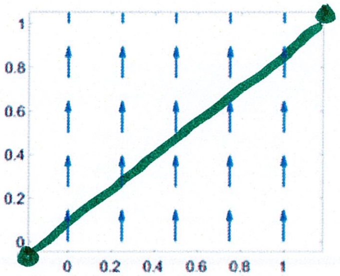
$\qquad (0,1) \qquad (1,0)$

$= \left[\frac{1}{2}(0)^2 + (0)(1) + c\right] - \left[\frac{1}{2}(1)^2 + (1)(0) + c\right]$

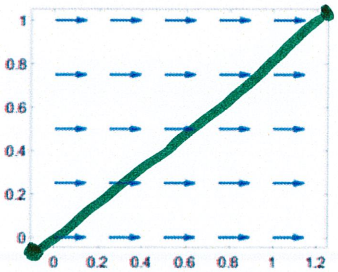
$= c - \left(\frac{1}{2} + c\right) = -\frac{1}{2}$

12. A particle is traveling on the path $y = x$ from $(0, 0)$ to $(1, 1)$. For which of the following force vector fields is the work done equal to 0?

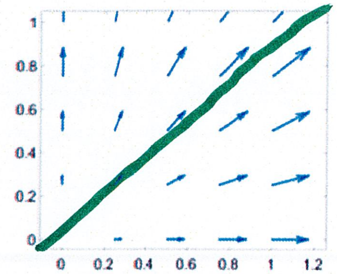
~~A~~



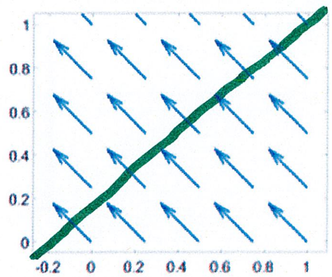
~~B~~



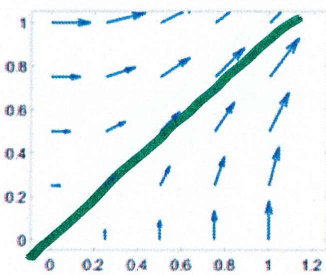
~~C~~



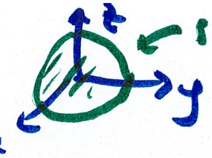
D



~~E~~



$W = \int_C \vec{F} \cdot d\vec{r}$ if $W = 0$, then $\vec{F} \cdot d\vec{r} = 0$ travel perpendicular to force surface integral in vector field → flux



surface of sphere only

Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ if $\mathbf{F}(x,y,z) = z\mathbf{k}$, S is the part of $x^2 + y^2 + z^2 = 4$ in the first octant, with outward-pointing normal vector.

Divergence Theorem does NOT apply directly because surface is open

- A. $\frac{8}{3}\pi$
- B. $\frac{4}{3}\pi$
- C. 2π
- D. 0
- E. $\frac{1}{2}\pi$

$S: \vec{r}(u,v) = \langle 2\sin u\cos v,\ 2\sin u\sin v,\ 2\cos u\rangle$

ϕ $\quad$ θ

$0 \le u \le \pi/2 \qquad 0 \le v \le \pi/2$

$\vec{r}_u = \langle 2\cos u\cos v,\ 2\cos u\sin v,\ -2\sin u\rangle$

$\vec{r}_v = \langle -2\sin u\sin v,\ 2\sin u\cos v,\ 0\rangle$

$\vec{r}_u \times \vec{r}_v = \langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ \underbrace{4\sin u\cos u}\rangle$

up ⟺ out in first octant

$\sin u \ge 0,\ \cos u \ge 0$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \langle 0, 0, 2\cos u\rangle \cdot \langle \quad \rangle\, du\,dv$$

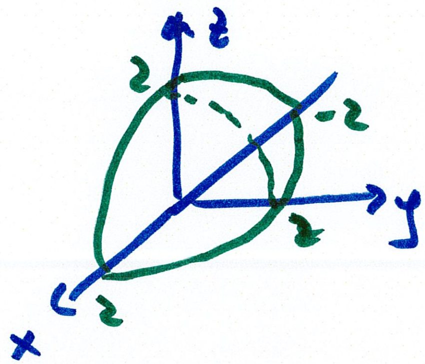
$$= \int_0^{\pi/2}\int_0^{\pi/2} 8\sin u\cos^2 u\, du\,dv \qquad w = \cos u \quad dw = -\sin u\,du$$

$$= \int_0^{\pi/2}\int_1^0 -8w^2\,dw\,dv = \frac{\pi}{2}\left(-\frac{8}{3}w^3\right)\Big|_1^0 = \frac{4\pi}{3}$$

Let $\mathbf{F} = \langle xy^2 + 1, yz^2 - x, x^2z + y \rangle$. Evaluate $\underbrace{\iint_S}_{\text{flux integral}} \mathbf{F} \cdot d\mathbf{S}$ where S is the boundary surface of the solid $D = \{(x, y, z) : \underbrace{x^2 + y^2 + z^2 \le 4}_{\text{sphere radius 2}}, y \ge 0, z \ge 0\}$ with an outward orientation.

right of xz-plane & above xy-plane

- ○ A. $\frac{16\pi}{5}$
- ○ B. 4π
- ○ C. $\frac{8\pi}{3}$
- ○ D. $\frac{32\pi}{5}$
- ○ E. $4\pi^2$



closed because $z \ge 0$, $y \ge 0$

Surface integral is still ok, but need 3 integrals (sphere, xy-plane, xz-plane)

plus, $\vec{F}$ is ugly

alternative: Divergence Theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \operatorname{div} \vec{F}\, dV$

$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \langle xy^2+1, yz^2-x, x^2z+y \rangle$

$= y^2 + z^2 + x^2 = \rho^2$ in spherical

$$\iiint_D \operatorname{div} \vec{F}\, dV = \underbrace{\int_0^{\pi}}_{\theta} \underbrace{\int_0^{\pi/2}}_{\phi} \underbrace{\int_0^2}_{\rho} \underbrace{\rho^2}_{\operatorname{div}\vec{F}}\ \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\pi} \int_0^{\pi/2} \int_0^{2} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

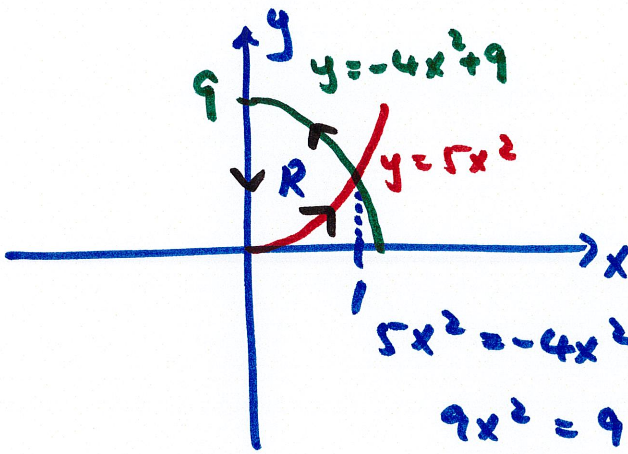
$$= \pi \int_0^{\pi/2} \frac{1}{5}\rho^5 \Big|_0^2 \sin\phi \, d\phi = \frac{32\pi}{5} \int_0^{\pi/2} \sin\phi \, d\phi = \frac{32\pi}{5} \left(-\cos\phi\right)\Big|_0^{\pi/2}$$

$$= \frac{32\pi}{5}$$

Compute the counterclockwise circulation of **F** around the closed curve C, where C is the region bounded above by $y = -4x^2 + 9$ and below by $y = 5x^2$ in the first quadrant.

$\mathbf{F} = (-5x + 4y)\mathbf{i} + (3x - 7y)\mathbf{j}$

- A. 8
- B. 11
- C. 24
- D. −6
- E. −14



$5x^2 = -4x^2+9$

$9x^2 = 9$

$x^2 = 1$

$x = 1$

want: $\oint_C \vec{F}\cdot d\vec{r}$

options: line integrals like first example today

or

Stokes' / Green's

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

$\oint_C f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$

where $\vec{F} = \langle f, g\rangle$

let's use Green's theorem

$\vec{F} = \langle \underbrace{-5x+4y}_{f}, \underbrace{3x-7y}_{g} \rangle$

$g_x - f_y = 3 - 4 = -1$

$R: \quad 0 \le x \le 1 \qquad 5x^2 \le y \le -4x^2+9$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^1 \int_{5x^2}^{-4x^2+9} (-1)\,dy\,dx = \int_0^1 (9x^2-9)\,dx = 3x^3 - 9x \Big|_0^1$

$= -6$